MASS 2023 Course:
Gravitation and Cosmology

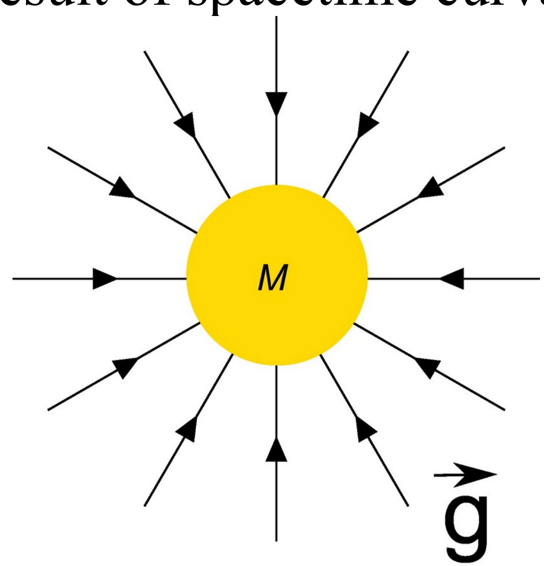
Predrag Jovanović
Astronomical Observatory Belgrade

Lecture 01

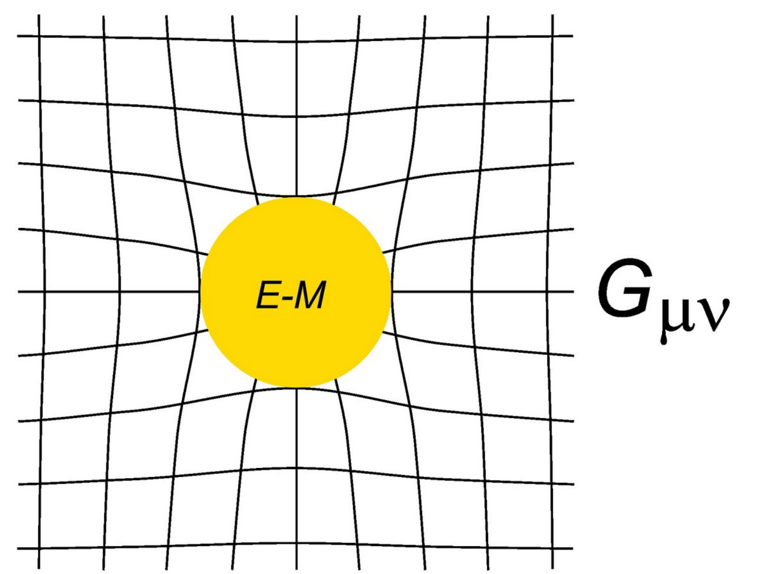
- Gravitational field
- Brief overview of Newton's theory of gravity
- General Relativity (GR) as a geometric theory of gravitation
- Concept of spacetime:
  - events
  - worldlines
- Einstein summation convention
- General coordinate transformations
- Some basic mathematical concepts:
  - Manifolds
- Exercises

Gravitational field

- In physics, a field is a physical quantity, represented by a scalar, vector or tensor, that has a value for each point in space and time
- A function assigning a scalar, vector, or tensor to each point in space and time:
  - scalar fields (temperature, humidity, pressure, ...)
  - vector fields (Newton's gravitational field, electric field, ...)
  - tensor fields (stress and strain in materials, Riemann curvature tensor, …)
- Gravitational field in Newton's theory of gravity is a vector field that exists in the space around every mass
- In contrast, gravity in General Relativity (GR) is not due to an effective force field, but a result of spacetime curvature



In Newton's theory of gravity, mass is the source of a gravitational field $\vec{g}$



In GR, mass-energy curves spacetime (Einstein tensor $G_{\mu\nu}$)

Newton's theory of gravity

- The gravitational force between two objects of masses M and m separated by a vector $\vec{r}$: $\vec{F}(\vec{r}) = -\frac{GMm}{r^2}\hat{\vec{r}}$
- The force acts on particle of mass m and gives it an acceleration according to Newton's second law: $\vec{F}(\vec{r}) = m\,\vec{g}(\vec{r})$
- Gravitational field is described by gravitational acceleration: $\vec{g}(\vec{r}) = -\frac{GM}{r^2}\hat{\vec{r}}$
- Equivalently, the gravitational potential Φ is related to the mass density ρ by Poisson's equation: $\nabla^2\Phi = 4\pi G\rho$
- The acceleration is then given by the gradient of the potential: $\vec{g}(\vec{r}) = -\nabla\Phi(\vec{r})$
- In the case of point mass M, Newton's gravitational potential is: $\Phi(r) = -\frac{GM}{r}$
- Gradient: $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$
- Laplace operator: $\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$ (divergence $\nabla\cdot$ of gradient ∇)
- Divergence of a vector $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$:

$$\mathrm{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

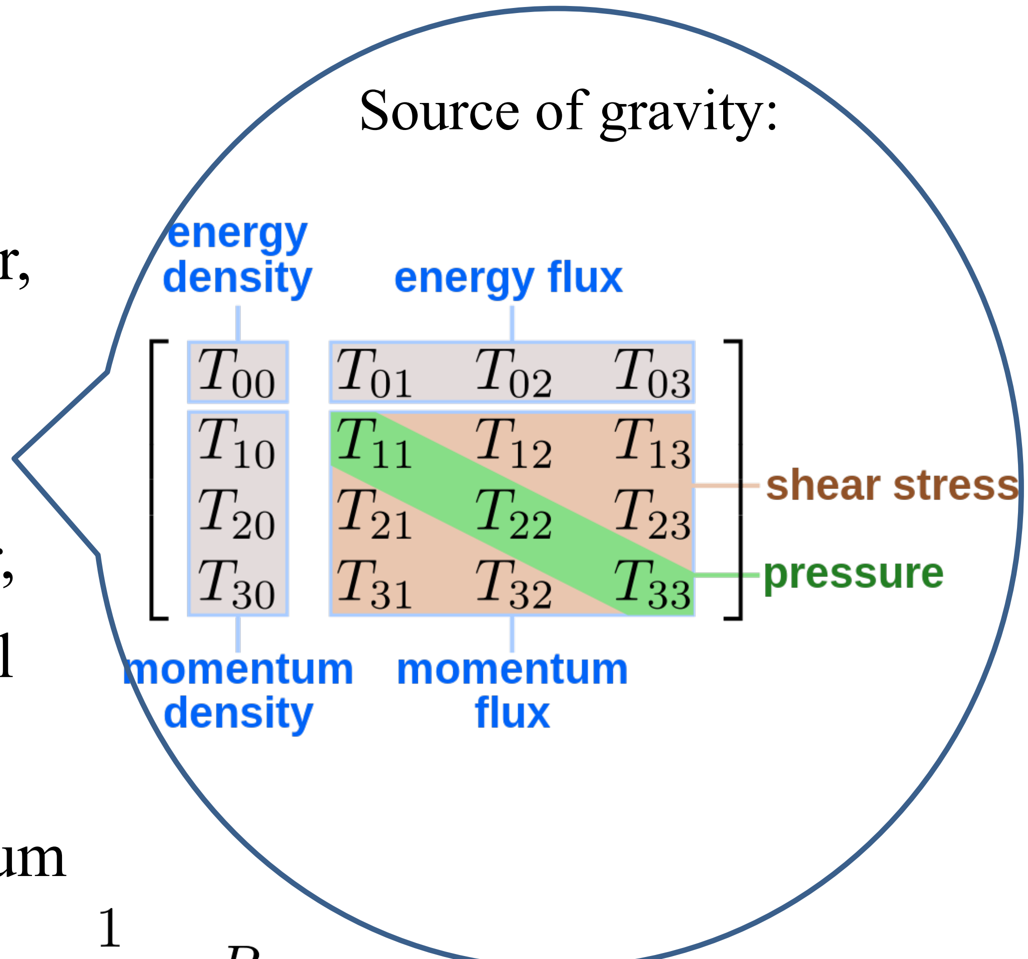
General Relativity (GR)

- GR is the standard theory of gravity, formulated by Albert Einstein in 1915
- It describes gravitation in terms of differential geometry of curved spacetime
- Fundamental interaction of gravitation as a result of spacetime being curved by matter and energy is described by field equations of GR:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where:

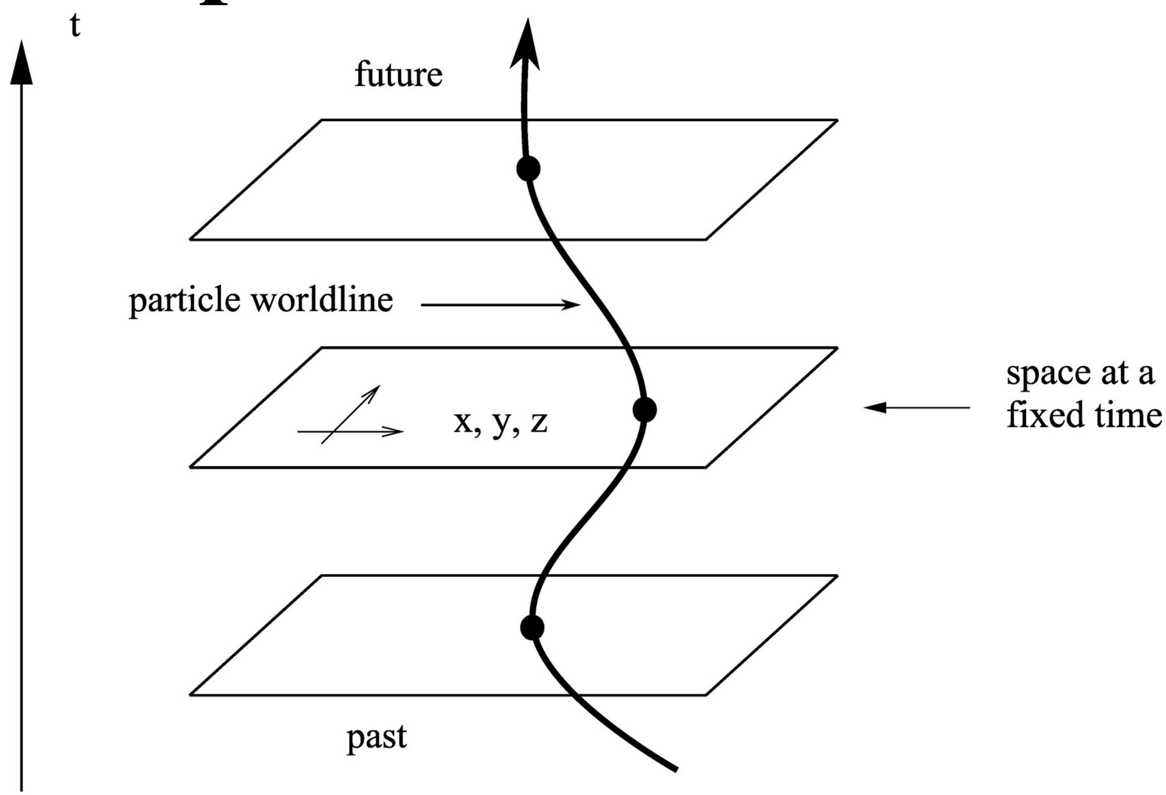
- $R_{\mu\nu}$ - Ricci curvature tensor,
- $g_{\mu\nu}$ - metric tensor,
- $T_{\mu\nu}$ - stress-energy tensor
- R - Ricci curvature scalar,
- G - Newton's gravitational constant,
- c - speed of light in vacuum



- Einstein tensor: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

Concept of spacetime

- **Spacetime** is a 4-dimensional set with elements labeled by 3 dimensions of space and 1 of time
- **Event** is an individual point in spacetime
- **Worldline** is a curve through spacetime (a parametrized 1-dimensional set of events), representing the path of a particle
- In Newtonian mechanics there is:



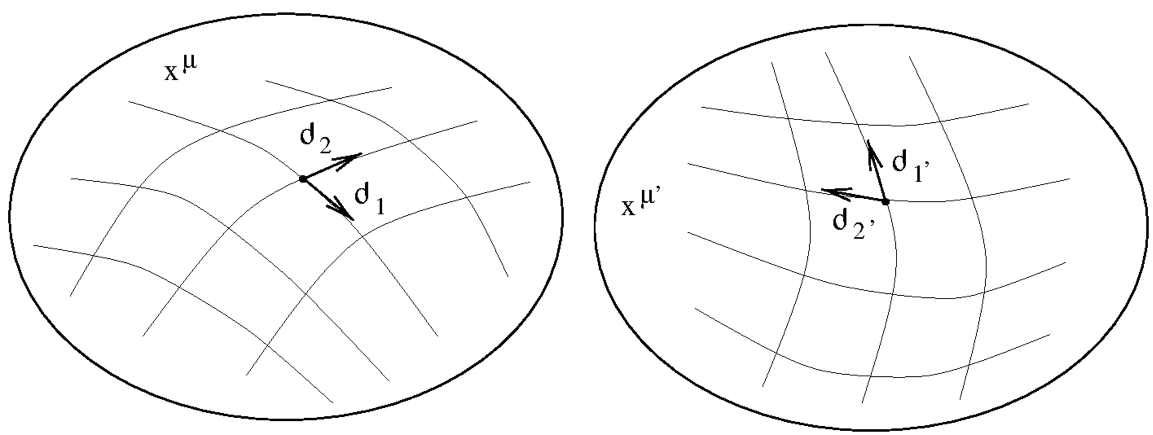
  - no limit on particle velocity
  - absolute division of spacetime into well-defined slices of space at a fixed moment in time
  - notion of **absolute simultaneity** of two events which occur at the same time
  - **absolute space:** the spatial interval between two simultaneous events is observer independent
  - **absolute time:** time interval between two events has absolute significance and all observers agree on its value
- **Euclidean geometry**: space interval Δl between two simultaneous events is given by Pythagorean theorem: $\Delta l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

Some important notation and conventions

- From now on we will use the following notation and convention:
  - Greek indices (α, β, …) run from 0 to 3
  - Lower-case latin indices (a, b, …) run from 1 to 3
  - **Einstein summation convention:** indices which appear both as superscripts and subscripts are summed over
- Example: $y = \sum_{i=1}^{3} c_i x^i = c_1 x^1 + c_2 x^2 + c_3 x^3$ is simplified to: $y = c_i x^i$
- The superscripts are not exponents but are indices of coordinates on spacetime, with 0 denoting the time coordinate: $x^\mu : \begin{array}{l} x^0 = ct \\ x^1 = x \\ x^2 = y \\ x^3 = z \end{array}$
- **General coordinate transformations** (which must be invertible) and their **inverse transformations**:

$$x^\mu \to x^{\mu'} = x^{\mu'}\left(x^0, x^1, x^2, x^3\right)$$

$$x^{\mu'} \to x^\mu = x^\mu\left(x^{0'}, x^{1'}, x^{2'}, x^{3'}\right)$$



- A quantity is **invariant** if its value is unchanged under coordinate transformations
- An equation is **covariant** if its form is unchanged under coordinate transformations

GR as a geometric theory of gravity

Physics	Geometry
• **When** and the **where** of a physical phenomenon constitute an **event.**	• An **event** is a **point** in a topological space
• The set of all events is a continuous space, named **spacetime**	• **Spacetime** is a **differentiable manifold** M
• **Gravitational phenomena** are manifestations of the **geometry** of spacetime	• **The gravitational field** is a **metric** g on M
• **Point-like particles** move in spacetime following special world-lines that are **"straight"**	• **Straight lines** are **geodesics**
• The **laws of physics** are the same for **all observers**	• **Field equations** are **generally covariant**

• Newtonian gravity vs. GR:

Spacetime = Differentiable Manifold

Gravitational Potentials = Metric

Forces = Connections

Some basic mathematical concepts

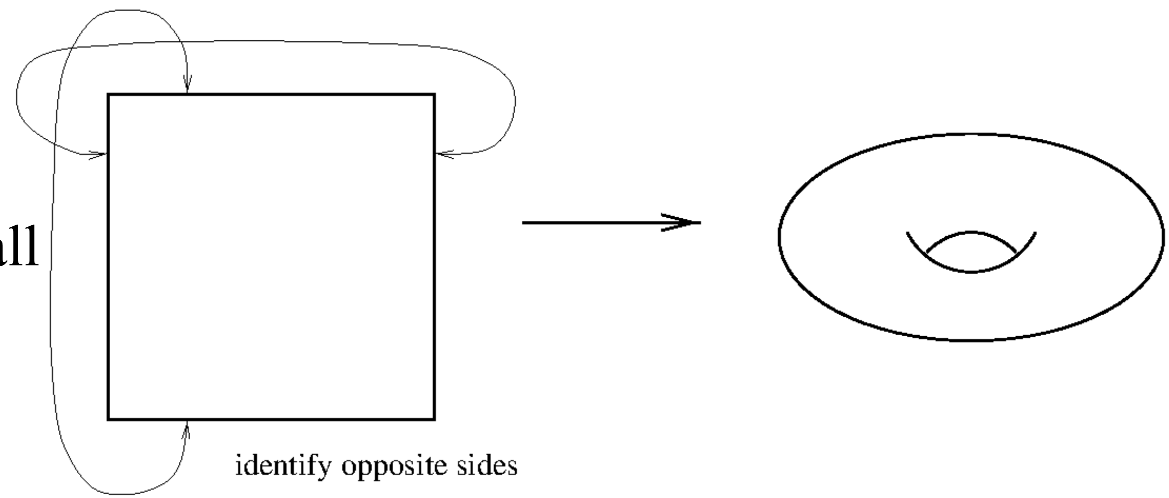
- **Mathematical space** is a set of elements called points with some additional features on the set (e.g. an operation, relation, metric or topology)
- **Topology** is set of neighborhoods for each point that satisfy some axioms of closeness
- **Topological space** is a mathematical space with topology (i.e. a mathematical space in which closeness is defined but cannot necessarily be measured by a numeric distance)
- Topological space is the most general type of a mathematical space that allows for the definition of limits, continuity, and connectedness
- Types of topological spaces: Euclidean spaces, metric spaces and manifolds
- n-dimensional **Euclidean space** is set $\mathbf{R}^n$ of all n-tuples of real numbers $(x^1, \ldots, x^n)$, equipped with the scalar (dot or inner) product of two vectors
- **Metric space** is an ordered pair (M, d) where M is a set of points and d is a function called **metric** on M which measures distance between the points.
  - A simple example is Euclidean space $\mathbf{R}^3$ equipped with Euclidean metric d:

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

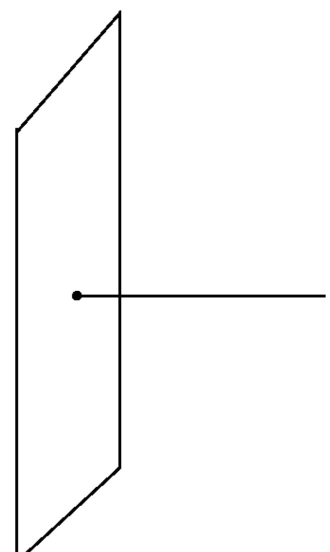
Manifolds

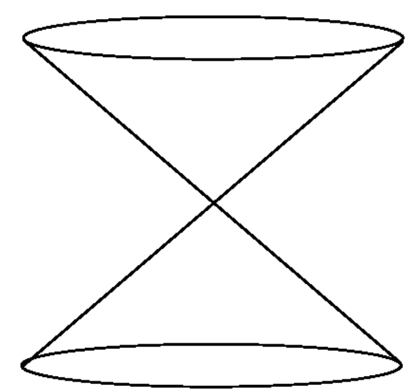
- **Manifold** (or differentiable manifold) is a topological space which may be curved and have a complicated topology, but locally looks like Euclidean space $\mathbf{R}^n$ near each point
- Manifold is constructed by smoothly joining together the Euclidean-like local regions

- Examples of manifolds:
  - Euclidean space $\mathbf{R}^n$
  - n-sphere S^n, since it is the locus of all points at some fixed distance from the origin in $\mathbf{R}^{n+1}$
  - The n-torus T^n, since it can be constructed from an n-dimensional cube by identifying its opposite sides



- These are not manifolds because somewhere they do not look locally like $\mathbf{R}^n$:
  - one-dimensional line running into a two-dimensional plane
  - two cones stuck together at their vertices





Exam question

1. Spacetime as differentiable manifold

Literature

- Textbook: Sean M. Carroll, 1997. Lecture Notes on General Relativity, arXiv, gr-qc/9712019

Exercise 1

- Write down $X_{ij}Y^{ij}$ explicitly as a sum of terms in $n = 2$ and $n = 3$ dimensions, respectively

Exercise 2

- Write down $X_iY^{ij}X_j$ explicitly as a sum of terms in $n = 2$ and $n = 3$ dimensions, respectively, assuming the following symmetry: $Y^{ij} = Y^{ji}$

Solution 1

- In $n = 2$ dimensions: $X_{ij}Y^{ij} = X_{11}Y^{11} + X_{12}Y^{12} + X_{21}Y^{21} + X_{22}Y^{22}$
- In $n = 3$ dimensions:

$$\begin{aligned} X_{ij}Y^{ij} &= X_{11}Y^{11} + X_{12}Y^{12} + X_{13}Y_{13} + X_{21}Y^{21} + X_{22}Y^{22} + \\ &+ X_{23}Y^{23} + X_{31}Y^{31} + X_{32}Y^{32} + X_{33}Y^{33} \end{aligned}$$

Solution 2

- In $n = 2$ dimensions: $X_i Y^{ij} X_j = Y^{11}(X_1)^2 + 2Y^{12}X_1X_2 + Y^{22}(X_2)^2$
- In $n = 3$ dimensions:

$$\begin{aligned} X_i Y^{ij} X_j &= Y^{11}(X_1)^2 + 2Y^{12}X_1X_2 + 2Y^{13}X_1X_3 + Y^{22}(X_2)^2 + \\ &+ 2Y^{23}X_2X_3 + Y^{33}(X_3)^2 \end{aligned}$$