MASS 2023 Course:
Gravitation and Cosmology

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Lecture 02

- Vectors and dual vectors
- Tangent space and coordinate basis
- Cotangent space and dual coordinate basis
- Contravariant and covariant transformation laws
- Tensors, their direct product and their transformation rules
- Exercises

Vectors

- In flat spaces, vectors are stretching from one point to another, and can be moved from point to point along a straight line joining these points
- In curved spaces, there is no unique way to move vectors from one point to another because there are no preferred curves joining these points
- **Vector** in a curved spacetime is just an object associated with a single point (i.e. each vector is located at a given point)
- **Basis** is any set of vectors which both spans the vector space (any vector is a linear combination of basis vectors) and is linearly independent (no basis vector is a linear combination of other basis vectors)
- Basis consists of the same number of vectors, known as the **dimension** of the space
- Vectors in four-dimensional spacetime are referred to as **four-vectors**
- If $\hat{e}_{(\mu)}$ is a basis, then any abstract vector A can be written as a linear combination of basis vectors: $A = A^{\mu}\hat{e}_{(\mu)}$, where the coefficients A^{μ} are the **components** of the vector A
- Often, the basis is entirely omitted and shorthand "the vector A^{μ}" is used
- Vectors are labeled with upper indices

Dual vectors

- **Vector space** is a set whose elements are vectors, together with the operations of vector addition and scalar multiplication
- If V and W are two vectors and a and b real numbers, then:
$$(a+b)(V+W) = aV + bV + aW + bW$$
- For every vector space, there is an associated vector space of equal dimension called **dual vector space**, and denoted by an asterisk
- Dual space is the space of all linear maps from the vector space to the real numbers
- Elements of dual vector space are called **dual vectors** or **one-forms**
- If ω is a dual vector, then: $\omega(aV + bW) = a\omega(V) + b\omega(W) \in \mathbf{R}$
- Also, if ω and η are dual vectors, then: $(a\omega + b\eta)(V) = a\omega(V) + b\eta(V)$
- For a given vector basis $\hat{e}_{(\mu)}$, a dual basis $\hat{\theta}^{(\nu)}$ satisfies: $\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \delta^{\nu}_{\mu}$,

  where δ^{ν}_{μ} is the **Kronecker delta** symbol: $\delta^{\nu}_{\mu} = \begin{cases} 0 & (\nu \neq \mu) \\ 1 & (\nu = \mu) \end{cases}$
- Every dual vector can be written in terms of its components, which are labeled with lower indices: $\omega = \omega_{\mu}\hat{\theta}^{(\mu)}$
- Action of a vector on a dual vector, and vice versa: $V(\omega) \equiv \omega(V) = \omega_{\mu}V^{\mu} \in \mathbf{R}$

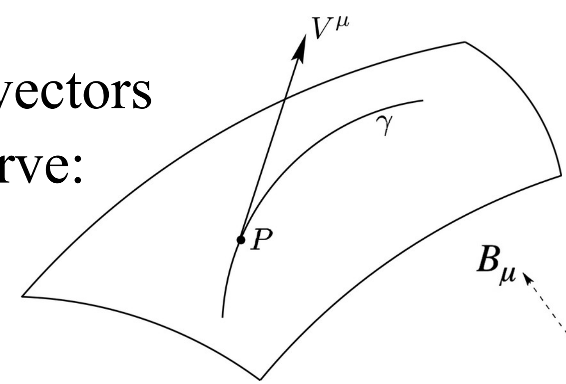
Examples of vectors and dual vectors

- For a space of n-component column vectors, the dual space is that of n-component row vectors, and the action is ordinary matrix multiplication

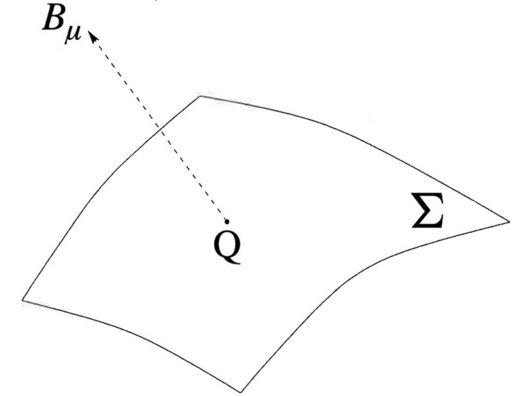
$$V = \begin{pmatrix} V^1 \\ V^2 \\ \vdots \\ V^n \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{pmatrix},$$

$$\omega(V) = \begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \\ \vdots \\ V^n \end{pmatrix} = \omega_i V^i$$

- **Parameterized curve** or **path** γ through spacetime is map $\gamma : \mathbf{R} \to M$ which is specified by the coordinates as a function $x^\mu(\lambda)$ of parameter λ along the curve
- **Canonical examples** of the vectors and dual vectors for the following two operators acting on a curve:
  - **partial derivatives**: ∂_μ
  - **gradient**: $\mathrm{d} = \dfrac{dx^\mu}{d\lambda}\partial_\mu$
- Partial derivatives result with the **tangent vector** $V(\lambda)$ to the curve $x^\mu(\lambda)$: $V = V^\mu \hat{e}_{(\mu)}, \; V^\mu = \dfrac{dx^\mu}{d\lambda}$
- In contrast to Euclidean space where gradient is an ordinary vector, in spacetime it is a dual vector



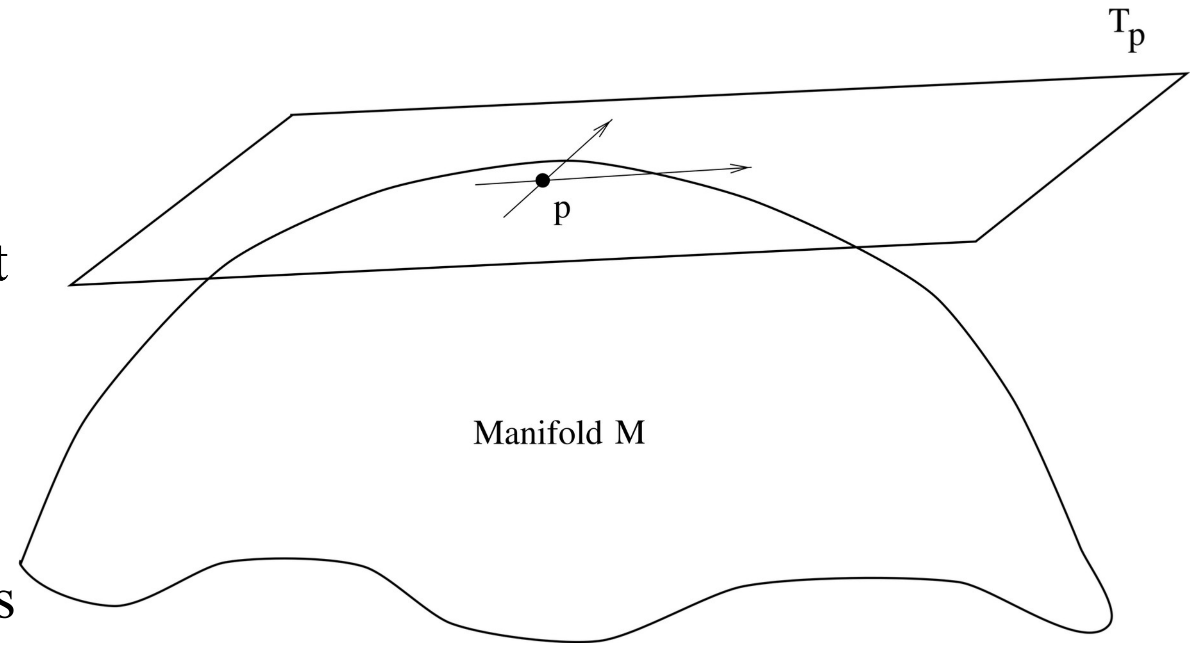
Vector V^μ as tangent to the curve γ



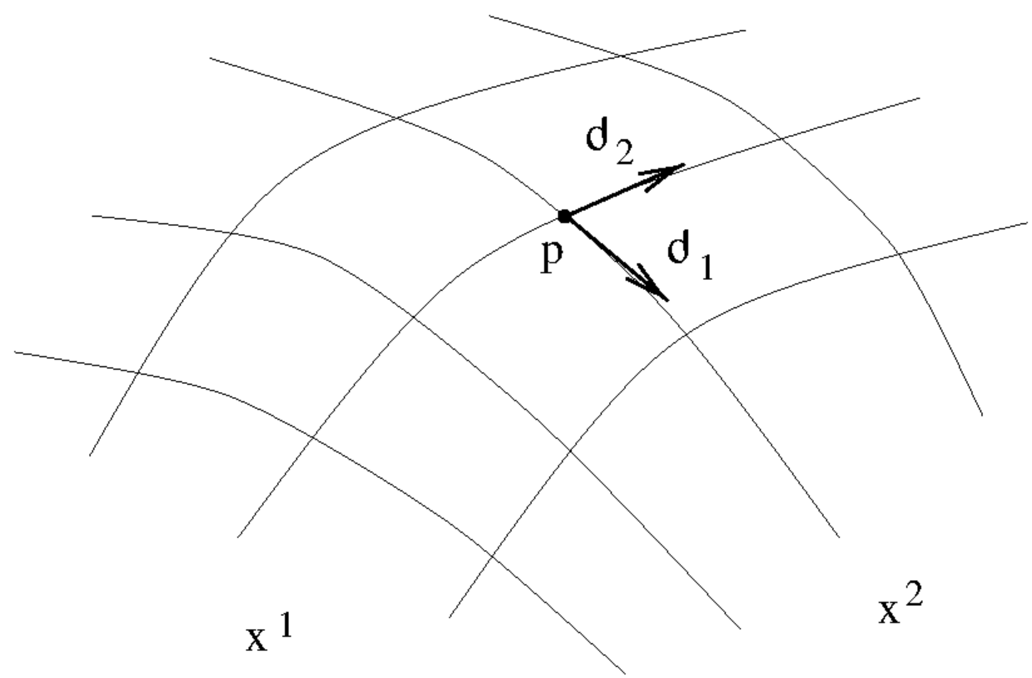
Dual vector B_μ as normal to the hypersurface Σ

Tangent space and coordinate basis

- **Tangent space** T_p at point p in a curved spacetime is the set of all possible vectors located at that point
- **Tangent bundle** $T(M)$ is set of all tangent spaces of a manifold M
- T_p is the space of all tangent vectors to the set of all parameterized curves γ through p

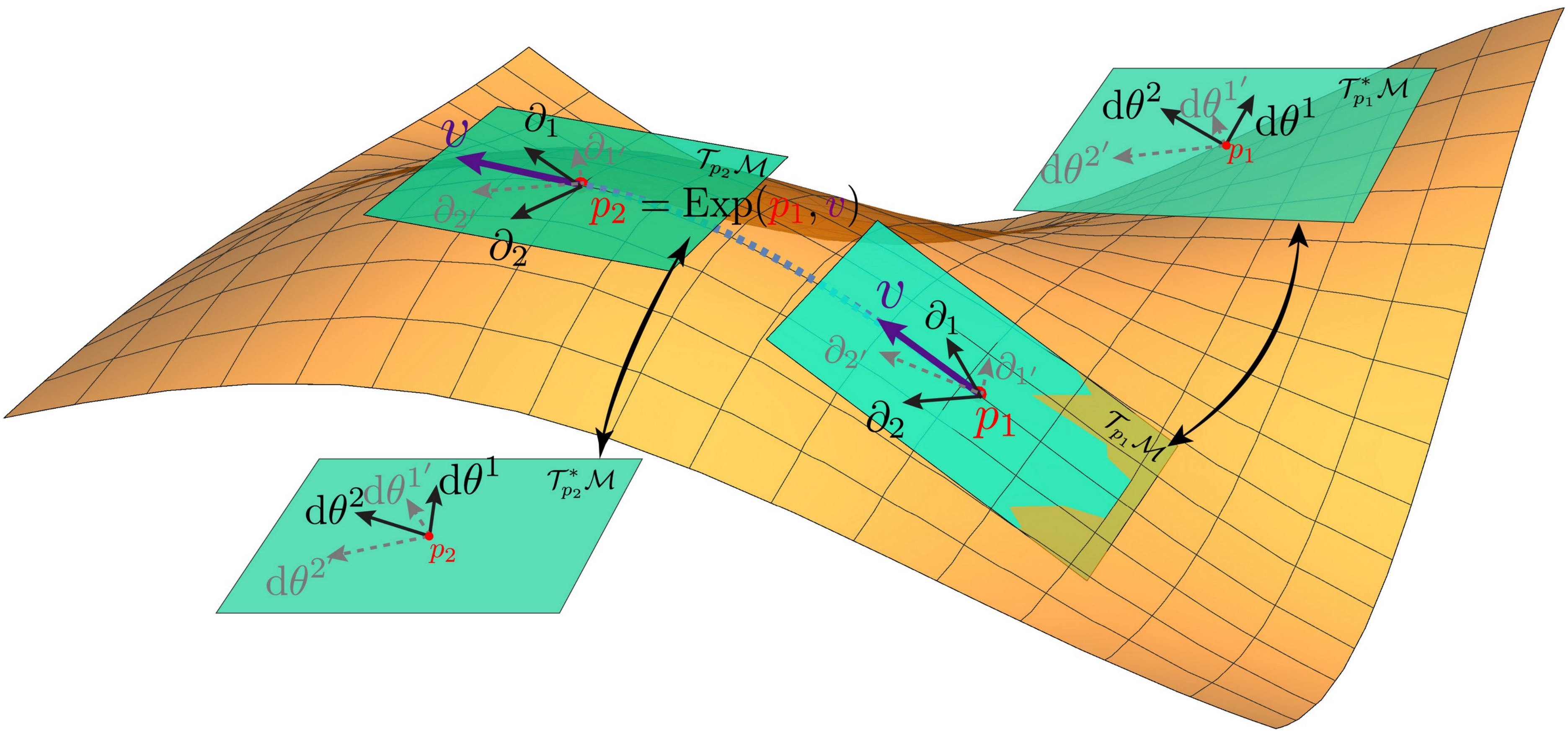


- Partial derivative operators $\{\partial_\mu\}$ at p form a particular basis $(\hat{e}_{(\mu)} = \partial_\mu)$ for T_p, known as **coordinate basis**, which consists of the basis vectors pointing along the coordinate axes



Cotangent space and dual coordinate basis

- Dual space to the tangent space T_p is called the **cotangent space** and denoted T_p^*
- **Cotangent bundle** $T^*(M)$ is set of all cotangent spaces over a manifold M
- Gradients $\{\mathrm{d}x^\mu\}$ of the coordinate functions provide **dual coordinate basis** for T_p^*
- An arbitrary dual vector is expanded into components as: $\omega = \omega_\mu \, \mathrm{d}x^\mu$



Tangent spaces with coordinate basis and cotangent spaces with dual coordinate basis

Contravariant and covariant transformation laws

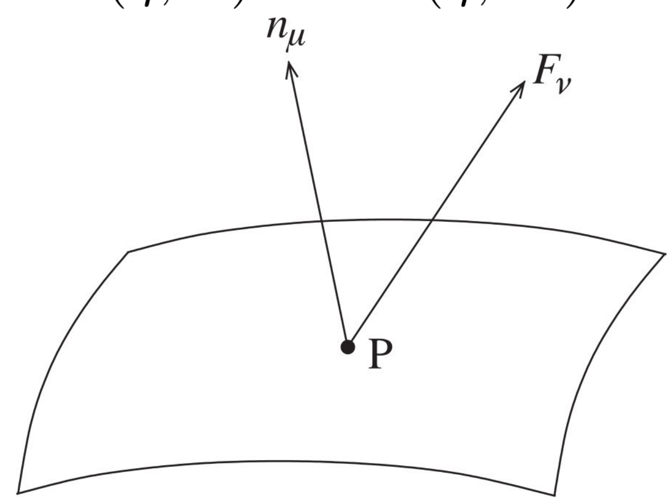
- **Contravariant vectors** (vectors with upper indices) are vectors from T_p
- **Covariant vectors** (vectors with lower indices) are dual vectors from T_p^*
- Since the basis contravariant vectors are $\hat{e}_{(\mu)} = \partial_\mu$, the corresponding basis contravariant vectors in some new coordinate system $x^{\mu'}$ are given by: $\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$
- From the demand that the vector $V = V^\mu \partial_\mu$ stays unchanged by a change of basis: $V^\mu \partial_\mu = V^{\mu'} \partial_{\mu'} = V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$, and since the matrix $\partial x^{\mu'} / \partial x^\mu$ is the inverse of the matrix $\partial x^\mu / \partial x^{\mu'}$ we get the following:
- **Contravariant transformation law**: $\boxed{V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu}$
- The basis dual vectors in a new coordinate system $x^{\mu'}$ are given as: $\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \mathrm{d}x^\mu$
- Components of dual vectors are transformed by the following:
- **Covariant transformation law:** $\boxed{\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu}$

Tensors

- **Tensor** is a straightforward generalization of vectors and dual vectors
- Tensor T of **type** (k, l) is a multilinear map from a collection of vectors and dual vectors to $\mathbf{R}$: $\underbrace{T_p \times \cdots \times T_p}_{k \text{ times}} \times \underbrace{T_p^* \times \cdots \times T_p^*}_{l \text{ times}} \to \mathbf{R}$, where × denotes the Cartesian product, so that e.g. $T_p \times T_p$ is the space of ordered pairs of vectors
- Multilinearity means that a tensor acts linearly in each of its arguments, e.g. for a tensor T of type (1, 1) this holds:

$$T(a\omega + b\eta, cV + dW) = acT(\omega, V) + adT(\omega, W) + bcT(\eta, V) + bdT(\eta, W)$$

- k is number of contravariant (upper) indices
- l is number of covariant (lower) indices
- A tensor is of **mixed type** if neither of k and l is 0
- **Rank** (or o**rder**) of a tensor is the total number of indices: $k + l$
- Tensors may have an arbitrary number of indices
- An n^{th}-rank tensor in m-dimensional space is a mathematical object that has n indices and m^n components



The second-rank stress tensor at P relates F_ν, the stress force at P, to the direction n_μ of the local normal to the surface at P

Direct product of two tensors

- **Direct product**: the product of the components of two tensors yields a tensor whose upper and lower indices consist of all the upper and lower indices of the two original tensors, and whose rank is equal to the sum of the two original ranks
- Direct product of two tensors is sometimes denoted by sign $\otimes$, but it is also common to simply write the tensors adjacent to each other and omit the $\otimes$ sign
  - Example 1: if $A^{\mu}{}_{\nu}$ and B^{ρ} are tensors, then $T^{\mu}{}_{\nu}{}^{\rho} \equiv A^{\mu}{}_{\nu} B^{\rho}$ is also a tensor
  - Example 2: direct product of two tensors *A* and *B* in matrix notation is:

$$
A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} = \begin{bmatrix}
a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1q} & \cdots & \cdots & a_{1n}b_{11} & a_{1n}b_{12} & \cdots & a_{1n}b_{1q} \\
a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{11}b_{2q} & \cdots & \cdots & a_{1n}b_{21} & a_{1n}b_{22} & \cdots & a_{1n}b_{2q} \\
\vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\
a_{11}b_{p1} & a_{11}b_{p2} & \cdots & a_{11}b_{pq} & \cdots & \cdots & a_{1n}b_{p1} & a_{1n}b_{p2} & \cdots & a_{1n}b_{pq} \\
\vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\
a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1q} & \cdots & \cdots & a_{mn}b_{11} & a_{mn}b_{12} & \cdots & a_{mn}b_{1q} \\
a_{m1}b_{21} & a_{m1}b_{22} & \cdots & a_{m1}b_{2q} & \cdots & \cdots & a_{mn}b_{21} & a_{mn}b_{22} & \cdots & a_{mn}b_{2q} \\
\vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\
a_{m1}b_{p1} & a_{m1}b_{p2} & \cdots & a_{m1}b_{pq} & \cdots & \cdots & a_{mn}b_{p1} & a_{mn}b_{p2} & \cdots & a_{mn}b_{pq}
\end{bmatrix}
$$

Tensor transformation rules

- Tensor with upper indices μ, ν, ... and lower indices κ, λ, ... transforms like the direct product of contravariant vectors U^μ W^ν ... and covariantvectors V_κ Y_λ …
- Tensor components obey the following transformation rules:

1) Zeroth-rank or type (0, 0) tensors or **scalars** are functions which remain invariant under the coordinate transformations: $\phi' = \phi'(x^{\mu'}) = \phi(x^\mu) = \phi$ (e.g. metric ds^2)
2) First-rank tensors (vectors) of type (1, 0): $T^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} T^\nu$ (contravariant vectors)

   and of type (0, 1): $T_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}} T_\nu$ (covariant vectors)
3) Second-rank tensors of type (2, 0): $T^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^\rho} \frac{\partial x^{\nu'}}{\partial x^\sigma} T^{\rho\sigma}$,

   of type (0, 2): $T_{\mu'\nu'} = \frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} T_{\rho\sigma}$,

   of type (1, 1): $T^{\mu'}_{\nu'} = \frac{\partial x^{\mu'}}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x^{\nu'}} T^\rho_\sigma$ (mixed tensor of rank 2)

- General tensor of type (n, m): $T^{\alpha' \ldots \beta'}_{\gamma' \ldots \delta'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \cdots \frac{\partial x^{\beta'}}{\partial x^\beta} \frac{\partial x^\gamma}{\partial x^{\gamma'}} \cdots \frac{\partial x^\delta}{\partial x^{\delta'}} T^{\alpha \ldots \beta}_{\gamma \ldots \delta}$
- Indices which are not summed over are called **free indices**, while indices which are summed over are called **dummy indices**

Exam questions

1. Vectors, dual vectors, tangent and cotangent space
2. Tensors, their direct product and their transformation rules

Literature

- Sean M. Carroll, 1997. *Lecture Notes on General Relativity*, arXiv, gr-qc/9712019

Exercise 1

- For a contravariant vector $V^\alpha = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and a covariant vector $X_\beta = [e, f, g, h]$ evaluate:

  a) their direct product: $T^\alpha_\beta = V^\alpha X_\beta$

  b) value of T^3_0 component of the direct product

Exercise 2

- A two index "object" $X^{\mu\nu}$ is defined by the "direct sum"of two vectors $X^{\mu\nu} = A^\mu + B^\nu$. Is $X^{\mu\nu}$ a tensor? Is there a transformation law to take X to a new coordinate system, i.e. to obtain $X^{\mu'\nu'}$ from $X^{\mu\nu}$?

Exercise 3

- Write down the transformation rules for all 3 types of second-rank tensors as the direct products of contravariant and covariant vectors

Exercise 4

- Write down the transformation rule for the tensor $T^{\mu}{}_{\nu}{}^{\lambda}$ as the direct product of contravariant and covariant vectors

Exercise 5

- Evaluate the following direct products of the tensors in the matrix form:

a) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}$

Solution 1

a) $T^{\alpha}_{\beta} = V^{\alpha} X_{\beta} = \begin{bmatrix} ae & af & ag & ah \\ be & bf & bg & bh \\ ce & cf & cg & ch \\ de & df & dg & dh \end{bmatrix}$

b) $T^{3}_{0} = V^{3} X_{0} = de$

Solution 2

If $X^{\mu\nu} = A^{\mu} + B^{\nu}$ and A^{μ} and B^{ν} are vectors, then this should hold:

$$X^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\rho}} A^{\rho} + \frac{\partial x^{\nu'}}{\partial x^{\sigma}} B^{\sigma}$$

It is not possible to express this as $X^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\rho}} \frac{\partial x^{\nu'}}{\partial x^{\sigma}} X^{\rho\sigma}$, and therefore there is no transformation law to take X to a new coordinate system and X cannot be a tensor

Solution 3

- Tensor of type (2, 0):

$$T^{\mu'\nu'} = A^{\mu'} B^{\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x^{\nu'}}{\partial x^{\beta}} B^{\beta} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} \frac{\partial x^{\nu'}}{\partial x^{\beta}} T^{\alpha\beta}$$

- Tensor of type (0, 2):

$$T_{\mu'\nu'} = A_{\mu'} B_{\nu'} = \frac{\partial x^{\alpha}}{\partial x^{\mu'}} A_{\alpha} \frac{\partial x^{\beta}}{\partial x^{\nu'}} B_{\beta} = \frac{\partial x^{\alpha}}{\partial x^{\mu'}} \frac{\partial x^{\beta}}{\partial x^{\nu'}} T_{\alpha\beta}$$

- Tensor of type (1, 1):

$$T^{\mu'}_{\nu'} = A^{\mu'} B_{\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x^{\beta}}{\partial x^{\nu'}} B_{\beta} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\nu'}} T^{\alpha}_{\beta}$$

Solution 4

$$T^{\mu'}{}_{\nu'}{}^{\lambda'} = A^{\mu'} B_{\nu'} C^{\lambda'} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x^{\beta}}{\partial x^{\nu'}} B_{\beta} \frac{\partial x^{\lambda'}}{\partial x^{\gamma}} C^{\gamma} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^{\gamma}} T^{\alpha}{}_{\beta}{}^{\gamma}$$

Solution 5

a) $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ a_{21}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1\times 0 & 1\times 5 & 2\times 0 & 2\times 5 \\ 1\times 6 & 1\times 7 & 2\times 6 & 2\times 7 \\ 3\times 0 & 3\times 5 & 4\times 0 & 4\times 5 \\ 3\times 6 & 3\times 7 & 4\times 6 & 4\times 7 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}$$