MASS 2023 Course:
Gravitation and Cosmology

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Lecture 03

- Spacetime as 4-dimensional pseudo-Riemannian manifold
  - Metric (fundamental) tensor
- Tensor algebra
  - Linear combination of tensors
  - Direct product of tensors (see the previous lecture)
  - Contraction of tensors
  - Raising and lowering indices of tensors
  - Scalar product of vectors
  - Inner product of two tensors
  - The quotient theorem
  - Symmetric and antisymmetric tensors
- Exercises

Spacetime as 4-dimensional manifold

- GR is a metric theory of gravitation in which 4-dimensional **pseudo-Riemannian manifold** *M* equipped with a **Riemannian metric** *g* is representing spacetime
- Pair (*M*, *g*) is the mathematical model of spacetime in GR
- Pseudo-Riemannian manifold is a differentiable manifold with a **metric tensor** $g_{\mu\nu}$ that is everywhere non-degenerate, meaning that the determinant $g = |g_{\mu\nu}|$ doesn't vanish and thus $g_{\mu\nu}$ is invertible
- **Covariant metric tensor** is a symmetric type (0, 2) tensor: $g_{\mu\nu} = g_{\nu\mu}$, and thus it has 10 independent components
- Spacetime interval *ds* between two events with a given infinitesimal coordinate separation is determined by the metric tensor $g_{\mu\nu}$: $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$
- ds^2 is invariant under general coordinate transformations:
$$ds'^2 = ds^2 \quad \Leftrightarrow \quad g'_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} dx^\mu dx^\nu$$
- **Contravariant metric tensor** $g^{\lambda\mu}$ is the inverse of $g_{\mu\nu}$, so that $g^{\lambda\mu} g_{\mu\nu} = \delta^\lambda_\nu$, where δ^μ_ν is the **Kronecker delta symbol**: $\delta^\mu_\nu = \begin{cases} 0 & (\nu \neq \mu) \\ 1 & (\nu = \mu) \end{cases}$
- δ^μ_ν is a mixed tensor, and aside from the scalars and zero (i.e. a tensor with all components equal to zero), δ^μ_ν (together with its direct products) is the only tensor whose components are the same in all coordinate systems

Metric (fundamental) tensor

- Metric tensor $g_{\mu\nu}$ is one of the most important features of curved space due to its numerous very significant roles, such as:
  - it supplies a notion of "past'" and "future"
  - it allows the computation of path length and proper time;
  - it determines the "shortest distance" between two points (and therefore the motion of test particles)
  - it replaces the Newtonian gravitational field
  - it provides a notion of locally inertial frames
  - it determines causality, by defining the speed of light faster than which no signal can travel
  - it replaces the traditional Euclidean three-dimensional dot product of Newtonian mechanics
- Any equation will be invariant under general coordinate transformations if it states the equality of two tensors with the same upper and lower indices
- In GR, equations are covariant under general coordinate transformations, an this is accomplished through the following algebraic operations with tensors: linear combination of tensors, direct product of tensors, contraction of tensors, raising and lowering indices of tensors, scalar product of two vectors, inner product of two tensors, symmetric and antisymmetric tensors

Tensor algebra I

- **Linear combination:** a linear combination of tensors with the same upper and lower indices is a tensor with these indices
- Example: let $A^{\mu}{}_{\nu}$ and $B^{\mu}{}_{\nu}$ be mixed tensors, and let $T^{\mu}{}_{\nu} \equiv aA^{\mu}{}_{\nu} + bB^{\mu}{}_{\nu}$, where a and b are scalars; then $T^{\mu}{}_{\nu}$ is also a tensor
- **Direct product** (reminder from the previous lecture): the product of the components of two tensors yields a tensor whose upper and lower indices consist of all the upper and lower indices of the two original tensors, and whose rank is equal to the sum of the two original ranks
- Example: if $A^{\mu}{}_{\nu}$ and B^{ρ} are tensors, then $T^{\mu}{}_{\nu}{}^{\rho} \equiv A^{\mu}{}_{\nu} B^{\rho}$ is also a tensor
- **Contraction**: Setting an upper and lower index equal and summing it over its four values yields a new tensor with these two indices absent and with rank reduced by 2
  - Example: if $T^{\mu}{}_{\nu}{}^{\rho\sigma}$ is a tensor, then $T^{\mu\rho} \equiv T^{\mu}{}_{\nu}{}^{\rho\nu}$ is also a tensor
- Tensor contraction is a generalization of **trace** in the sense that the trace is the simplest type of tensor contraction, namely a second-rank tensor contraction:
  - Example: trace of a mixed second-rank tensor T is a scalar: $\operatorname{tr} T_{\mu}^{\nu} = T_{\mu}^{\mu}$

Raising and lowering indices

- An operation obtained by combining the pervious three operations
- **Lowering indices**: if we take the direct product of a contravariant or mixed tensor T with the metric tensor $g_{\mu\nu}$, and contract the index μ with one of the contravariant indices of T, we get a new tensor in which this contravariant index is replaced by a covariant index ν
- Example: if $T^{\mu\rho}{}_{\sigma}$ is a tensor then $S_{\nu}{}^{\rho}{}_{\sigma} \equiv g_{\mu\nu} T^{\mu\rho}{}_{\sigma}$ is also a tensor
- **Raising indices**: if we take the direct product of a covariant or mixed tensor T with the inverse metric tensor $g^{\mu\nu}$, and contract the index μ with one of the covariant indices of T, we get a new tensor in which this covariant index is replaced by a contravariant index ν
- Example: if $S_{\mu}{}^{\rho}{}_{\sigma}$ is a tensor then $R^{\nu\rho}{}_{\sigma} \equiv g^{\mu\nu} S_{\mu}{}^{\rho}{}_{\sigma}$ is also a tensor
- Lowering an index and then raising it again gives back the original tensor
- Tensors obtained by raising and lowering indices are called **associated tensors** and are physically equivalent
- Tensor obtained by raising one index on the metric tensor $g_{\mu\nu}$ or by lowering one index on the inverse metric tensor $g^{\mu\nu}$, is the **Kronecker tensor**: $g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}{}_{\nu}$
- Raising both indices on $g_{\mu\nu}$ gives the inverse tensor $g^{\lambda\mu} g^{\kappa\nu} g_{\mu\nu} = g^{\lambda\kappa}$ and lowering both indices on $g^{\lambda\kappa}$ gives the metric tensor $g_{\mu\nu}$

Tensor algebra II

- Contraction on a pair of indices that are either both contravariant or both covariant is not possible in general, but it is possible in the presence of the metric tensor $g_{\mu\nu}$
- **Metric contraction**: a combined operation of using the metric tensor to raise or lower one of the indices, as needed, followed by the usual operation of contraction
- **Scalar** (**inner** or **dot**) product of two vectors is:
$$A \cdot B = (A^\mu, B^\nu) = g_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu,$$
where the final result was obtained by lowering the contravariant index ν
- Scalar product of two vectors is a scalar and if it is equal to 0, then the two vectors are referred to as **orthogonal**
- The **square** of a vector is scalar product of the vector with itself
- **Inner product** of two tensors (a generalization of the scalar product of vectors) is obtained by taking the direct product of two tensors for the special case where one index is repeated, and taking the sum over this repeated index (contraction)
- Resulting tensor that has rank equal to the sum of the original ranks reduced by 2 for one contraction
- **The quotient theorem** (criterion for tensor character): if the result of taking the product (direct or inner) of a given set of elements with a tensor of any specified type and arbitrary components is known to be a tensor, then the given elements are the components of a tensor

Symmetric and antisymmetric tensors

- Tensor is **symmetric** in any of its indices if it is unchanged under exchange of those indices
  - Example 1: if $S_{\mu\nu\rho} = S_{\nu\mu\rho}$ then $S_{\mu\nu\rho}$ is symmetric in its first two indices
  - Example 2: if $S_{\mu\nu\rho} = S_{\mu\rho\nu} = S_{\rho\mu\nu} = S_{\nu\mu\rho} = S_{\nu\rho\mu} = S_{\rho\nu\mu}$ then $S_{\mu\nu\rho}$ is symmetric in all three of its indices
- Tensor is **antisymmetric** in any of its indices if it changes sign when those indices are exchanged
  - Example: if $A_{\mu\nu\rho} = -A_{\rho\nu\mu}$ then $A_{\mu\nu\rho}$ is antisymmetric in its first and third indices (or simply antisymmetric in μ and ρ)
- Tensor which is (anti-) symmetric in all of its indices, is referred to as **completely (anti-) symmetric**
- Examples:
  - Metric and the inverse metric tensors are symmetric
  - **Levi-Civita tensor** $\epsilon_{\mu\nu\rho\sigma}$ is completely antisymmetric:

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$$

Exam questions

1. Spacetime as 4-dimensional pseudo-Riemannian manifold and metric tensor
2. Tensor algebra

Literature

- Textbook: Weinberg, S., 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, Wiley-VCH

Exercise 1

- Calculate the components of covariant metric tensor $g_{\mu\nu}$ and contravariant (inverse) metric tensor $g^{\mu\nu}$ for:
  a) two-dimensional flat Euclidean space (Euclidean plane) in polar coordinates, with metric: $ds^2 = dr^2 + r^2 d\theta^2$
  b) two-sphere metric: $ds^2 = a^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$

Exercise 2

- Use the fact that under a coordinate transformation $x^\mu \to x'^\mu$ coordinate differential transforms as a contravariant vector due to rules of partial differentiation: $dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$
  to prove that $g_{\mu\nu}$ is indeed a covariant tensor

Exercise 3

- Show that the determinant of the metric tensor $g = \det(g_{\mu\nu})$ is not a scalar

Exercise 4

- For the two tensors $V_{\varepsilon}^{\mu\nu}$ and $W_{\alpha\beta}^{\gamma}$ evaluate their:
  a) direct product
  b) inner product for $\varepsilon = \gamma$
  c) inner product for $\varepsilon = \gamma$ and $\alpha = \nu$

Exercise 5

- Find the covariant vector A_μ associated to the contravariant vector $\mathrm{A}^\mu = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$,

  as well as its square A^2 in Minkowski spacetime where: $\mathrm{g}_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 6

- Write the general forms of a symmetric tensor $S_{\mu\nu} = S_{\nu\mu}$ and an asymmetric tensor $A_{\mu\nu} = -A_{\nu\mu}$ as matrices of latin letters *a, b, c, …*

Exercise 7

- Show that a second rank tensor F which is antisymmetric in one coordinate frame ($F_{\mu\nu} = -F_{\nu\mu}$) is antisymmetric in all frames. Show that the contravariant components are also antisymmetric ($F^{\mu\nu} = -F^{\nu\mu}$)

Solution 1

a)

$$g_{rr} = 1, \quad g_{r\theta} = 0, \quad g_{\theta r} = 0, \quad g_{\theta\theta} = r^2 \quad \Rightarrow \quad g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

$$g^{rr} = 1, \quad g^{r\theta} = 0, \quad g^{\theta r} = 0, \quad g^{\theta\theta} = r^{-2} \quad \Rightarrow \quad g^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}$$

b)

$$g_{\theta\theta} = a^2, \quad g_{\theta\phi} = 0, \quad g_{\phi\theta} = 0, \quad g_{\phi\phi} = a^2 \sin^2\theta \Rightarrow g_{\mu\nu} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{bmatrix}$$

$$g^{\theta\theta} = a^{-2}, \quad g^{\theta\phi} = 0, \quad g^{\phi\theta} = 0, \quad g^{\phi\phi} = a^{-2} \sin^{-2}\theta \Rightarrow g^{\mu\nu} = \begin{bmatrix} a^{-2} & 0 \\ 0 & a^{-2} \sin^{-2}\theta \end{bmatrix}$$

Solution 2

$$ds'^2 = g'_{\mu\nu} dx'^\mu dx'^\nu = g'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} dx^\alpha \frac{\partial x'^\nu}{\partial x^\beta} dx^\beta = g_{\alpha\beta} dx^\alpha dx^\beta = ds^2 \quad \Rightarrow$$

$$\Rightarrow \quad g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

Solution 3

Under the coordinate transformation $x^{\mu} \rightarrow x'^{\mu}$ the transformation of $g_{\alpha\beta}$ is

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$$

and therefore the transformation of g is

$$g' = \det(g'_{\mu\nu}) = \det\left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\right) \det\left(\frac{\partial x^{\beta}}{\partial x'^{\nu}}\right) \det(g_{\alpha\beta}) = g\left[\det\left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\right)\right]^2 \neq g$$

Since $g' \neq g$, g is not a scalar.

Solution 4

a) $V_{\varepsilon}^{\mu\nu} W_{\alpha\beta}^{\gamma} = U_{\varepsilon\alpha\beta}^{\mu\nu\gamma}$

b) $V_{\varepsilon}^{\mu\nu} W_{\alpha\beta}^{\varepsilon} = U_{\alpha\beta}^{\mu\nu}$

c) $V_{\varepsilon}^{\mu\nu} W_{\nu\beta}^{\varepsilon} = U_{\beta}^{\mu}$

Solution 5

$$A_\mu = g_{\mu\nu} A^\nu = \begin{bmatrix} -1, & -1, & 2, & 3 \end{bmatrix}$$

$$A^2 = -(1)^2 + (-1)^2 + 2^2 + 3^2 = 13$$

Solution 6

$$S_{\mu\nu} = S_{\nu\mu} = \begin{bmatrix} w & a & b & c \\ a & x & d & e \\ b & d & y & f \\ c & e & f & z \end{bmatrix}, \qquad A_{\mu\nu} = -A_{\nu\mu} = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$

Solution 7

$$F'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}F_{\alpha\beta} = -\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}F_{\beta\alpha} = -\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x^\alpha}{\partial x'^\nu}F_{\alpha\beta} = -F'_{\nu\mu}$$

$$F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} = -g^{\mu\alpha}g^{\nu\beta}F_{\beta\alpha} = -g^{\mu\beta}g^{\nu\alpha}F_{\alpha\beta} = -F^{\nu\mu}$$