MASS 2023 Course:

Gravitation and Cosmology

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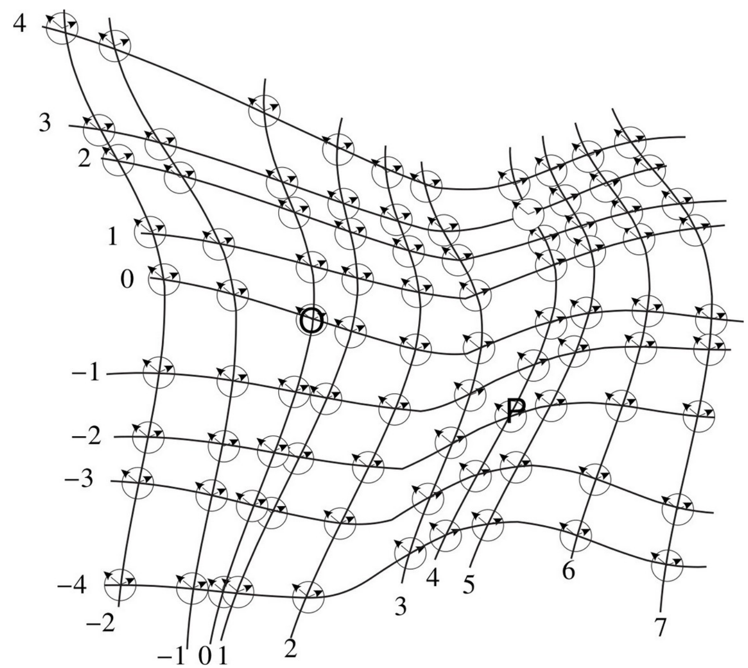
Lecture 04

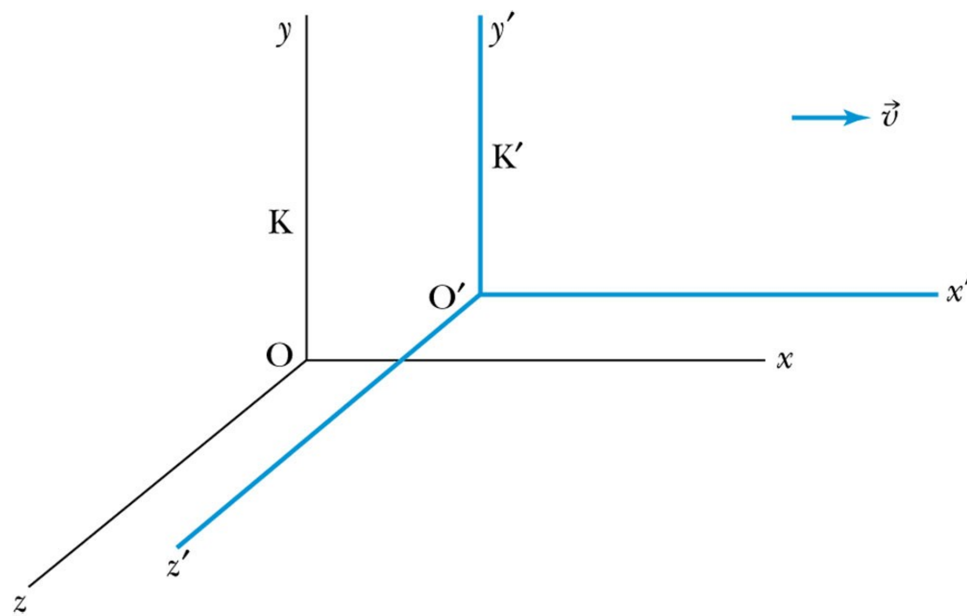
- The concept of relativity
  - Reference frames (inertial and non-inertial)
  - Newtonian (classical) relativity and Galilean transformations
  - Special Relativity (SR) and its postulates (principle of relativity and constancy of the speed of light)
- Spacetime in absence of gravity
  - Minkowski spacetime, light cone and spacetime intervals
  - Lorentz transformations
- Tensor formulation of SR
  - Lorentz transformations in tensor form
  - Four-vectors of velocity and energy-momentum
- Energy-momentum tensor
  - Perfect fluid
  - Law of energy-momentum conservation
- Exercises

Reference frame

- A **reference frame** consists of an abstract coordinate system and the set of physical reference points that uniquely fix the coordinate system and standardize measurements within that frame
- **Inertial reference frame** is a frame where Newton's laws hold true (i.e. if no external force is acting on a body it will stay at rest or remain in uniform motion)
- In Newtonian mechanics, inertial frames are at rest or in a state of uniform motion with respect to absolute space
- **Non-inertial reference frame** is a frame that is accelerated with respect to the assumed inertial reference frame



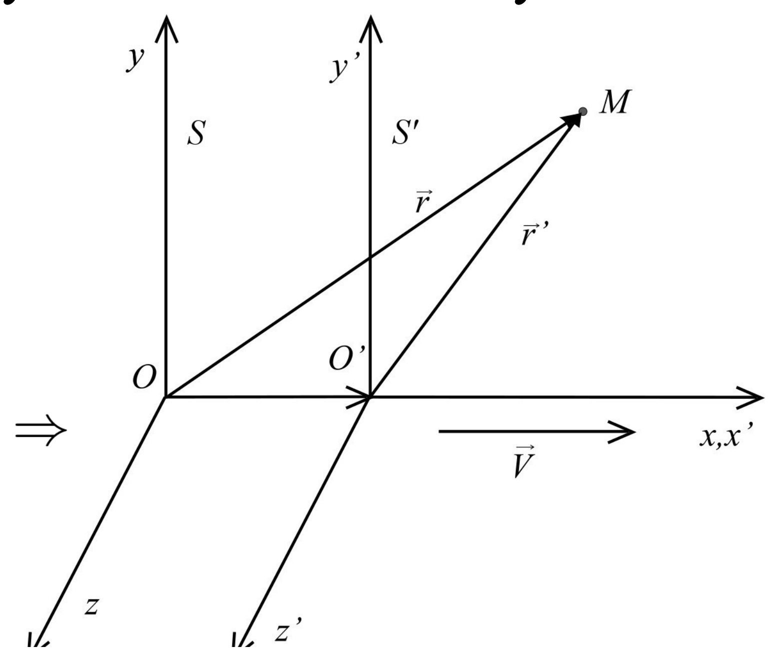
Reference frame as set of reference points measuring the position of events and the time at which they occur



Inertial reference frames K and K'

Newtonian (classical) Relativity

- Newton's laws of motion must be measured with respect to (relative to) some reference frame
- Hypothetical **luminiferous aether** was considered as medium for the propagation of light, as well as the **absolute reference frame**
- **Newtonian principle of relativity** or **Galilean invariance**:
  If Newton’s laws are valid in one reference frame, then they are also valid in another reference frame moving at a uniform velocity relative to the first system
- **Galilean Transformations (GT):**
- Relation between the coordinates $M(x, y, z)$ and $M(x', y', z')$ of point M in inertial frames S and S' at instant of time $t = t'$



$$O\vec{M} = O\vec{O}' + O'\vec{M} \wedge O\vec{O}' = \vec{V}t' \Rightarrow \vec{r} = \vec{r}' + \vec{V}t' \Rightarrow$$

$$x = x' + Vt', \quad y = y', \quad z = z', \quad t = t'$$

- **Inverse GT:**

$$\vec{r}' = \vec{r} - \vec{V}t \quad \Rightarrow \quad x' = x - Vt, \quad y' = y, \quad z' = z, \quad t' = t$$

- **Classical velocity addition law:** $\vec{v} = \vec{v}\,' + \vec{V}$

Basics of Special Relativity (SR)

- Modern theory of relativity is formulated by Albert Einstein and it consists of:
  - **Special Relativity (SR)** which considers flat spacetime as a special case of a 4-dimensional manifold, known as **Minkowski space,** and studies relative motions in inertial frames, without considering acceleration and gravity
  - **General Relativity (GR)** which considers curved spacetimes as 4-dimensional **Riemannian manifolds** and studies accelerated relative motions and gravity
- In 1905, Einstein proposed the following **postulates** of SR:
  1) **The principle of relativity:** the laws of physics are the same in all inertial systems. There is no way to detect absolute motion, and no preferred inertial system exists
  2) **The constancy of the speed of light:** observers in all inertial systems measure the same value for the speed of light in a vacuum
- The second postulate accounts for the observation by Michelson and Morley that aether does not exist and that the speed of light is the same in all inertial systems
- Positions in spacetime are not absolute (i.e. there is no absolute reference frame), but they depend on the observer's location and velocity
- Set of allowed paths that particles can take in SR is essentially different than in Newtonian mechanics due to the absence of a preferred time-slicing

Spacetime in absence of gravity: Minkowski spacetime

- Spacetime in SR is called **Minkowski spacetime**, and it is a special case of 4-dimensional manifold equipped with **Minkowski metric tensor** $\eta_{\mu\nu}$, having metric signature $(-+++)$ and elements defined by the matrix:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Minkowski spacetime is flat because all metric coefficients are constant and metric is real and pseudo-Euclidean, since it could not be transformed into Euclidean by any real coordinate transformations

- An element of spacetime interval *ds* in SR: $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad x^\mu :$

$$x^0 = ct \quad x^1 = x \quad x^2 = y \quad x^3 = z$$

- **Spacetime interval** Δ*s* between two events:

$$(\Delta s)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

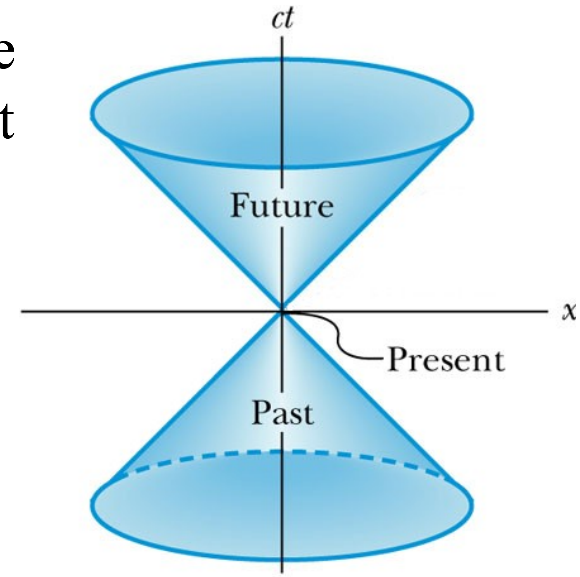
- Spacetime interval is invariant under changes of inertial coordinates:

$$(\Delta s)^2 = -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2,$$

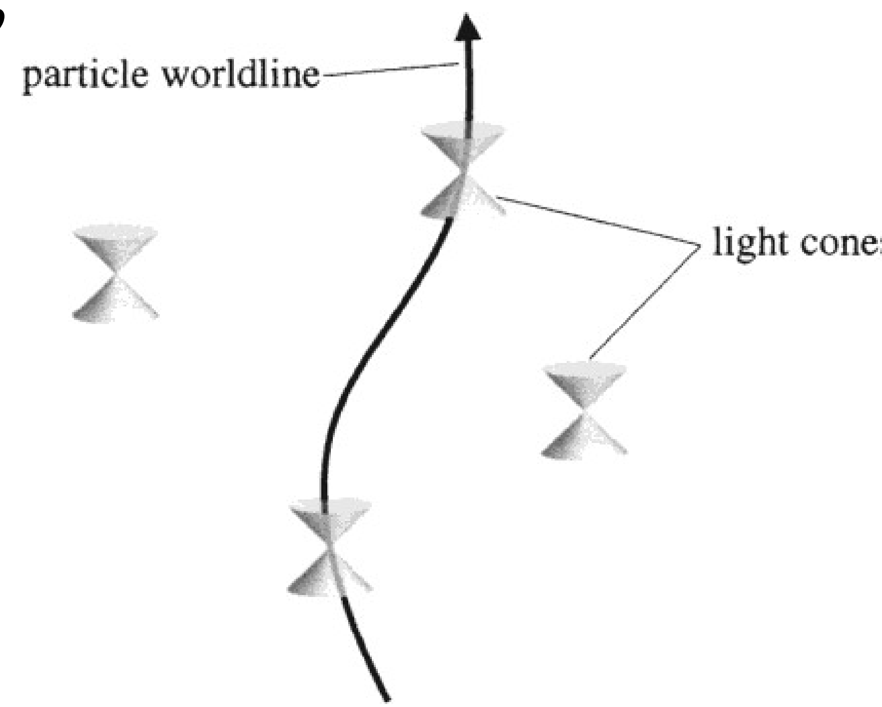
where (ct, x, y, z) and (ct', x', y', z') are spacetime coordinates in two inertial frames (**inertial coordinates**)

Spacetime in SR

- At any event it is possible to define a **light cone**, which is the locus of paths through spacetime that could be taken by light rays passing through that event: $x^2 + y^2 + z^2 = c^2t^2$
- Physical particles cannot travel faster than light, and thus move along paths that always remain inside the light cones
- The set of all events inside the future and past light cones of an event *p* are called **timelike separated** from *p*, while those outside the light cones are **spacelike separated** and those on the cones are **lightlike** or **null separated** from *p*



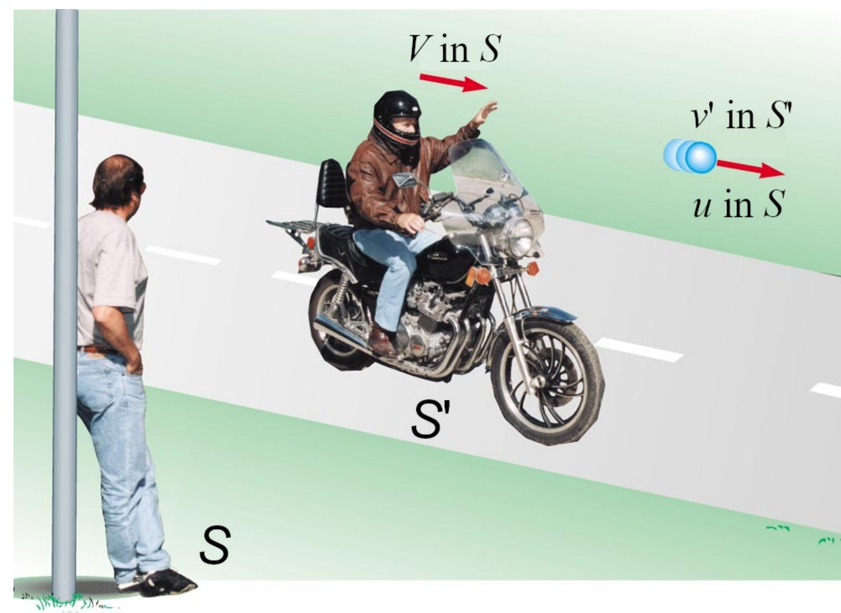
- There is no well-defined notion of two separated events occurring "at the same time"
- **Coordinate time** *t* between two events is measured using the observer's clock
- **Proper time** τ along a timelike worldline is measured by a clock following that worldline
- Proper time is independent of coordinates and invariant under Lorentz transformations
- Relation between the proper time and spacetime interval: $(c\Delta\tau)^2 = -(\Delta s)^2 = -\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$



Lorentz Transformations (LT)

- The special set of linear transformations that:
  - preserve the constancy of the speed of light c between inertial observers
  - account for the problem of simultaneity between these observers (in contrast to classical mechanics, in SR the events considered simultaneous in one inertial frame may not be simultaneous in other inertial frame, since the observers have their own clocks and meter sticks)
- Relation between the coordinates of an event (t, x, y, z) in the inertial frame S and its coordinates (t', x', y', z') in the inertial frame S', moving with constant velocity V with respect to S along their common x-axis (**"boost"**):

$$t' = \gamma\left(t - \frac{Vx}{c^2}\right),\ x' = \gamma\left(x - Vt\right),\ y' = y,\ z' = z$$



- **Lorentz factor:** $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{V^2}{c^2}}}$
- **Inverse LT** (by substituting V with -V)**:**

$$t = \gamma\left(t' + \frac{V}{c^2}x'\right),\ x = \gamma\left(x' + Vt'\right),\ y = y',\ z = z'$$

- For $V \ll c$ LT reduce to GT (correspondence principle)
- **Relativistic velocity addition law** for collinear motions: $u = \dfrac{V + v'}{1 + (Vv'/c^2)}$

Tensor formulation of SR

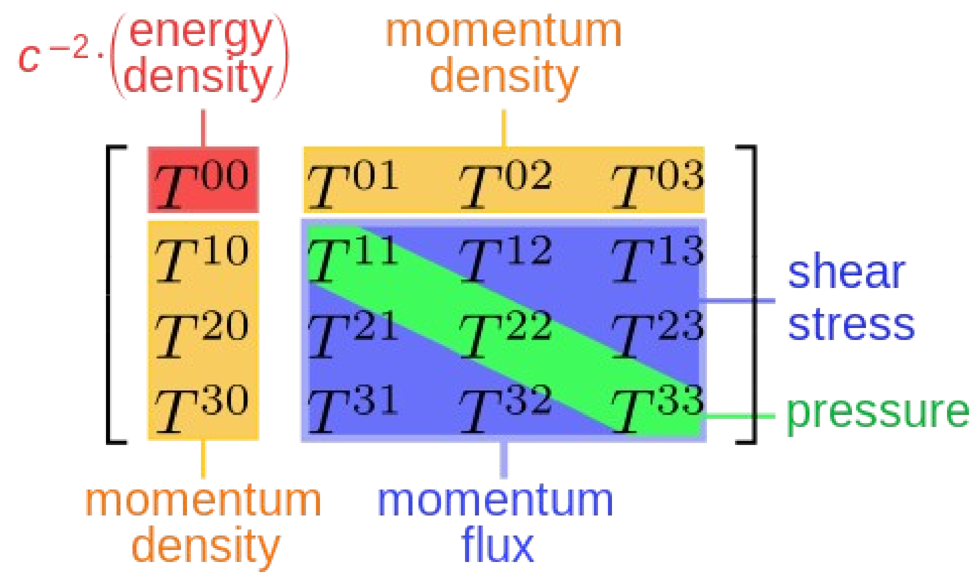
- LT in tensor notation: $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$, and in matrix notation:

$$x' = \Lambda\, x \quad \Leftrightarrow \quad \begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -V\gamma/c & 0 & 0 \\ -V\gamma/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

- Inverse LT in matrix notation: $x = \Lambda^{-1} x'$, where $\Lambda^{-1}(V) = \Lambda(-V)$, and in tensor notation: $x^{\mu} = \Lambda_{\nu}{}^{\mu}\, x'^{\nu}$, where $\Lambda_{\nu}{}^{\mu}$ is the inverse of $\Lambda^{\mu}{}_{\nu}$: $\Lambda_{\alpha}{}^{\gamma} \Lambda^{\alpha}{}_{\beta} = \delta^{\gamma}{}_{\beta}$
- Vectors in Minkowski spacetime are called **four-vectors** and then: $A_{\mu} = \eta_{\mu\nu}\, A^{\nu}$
- 4-vectors with positive, negative or zero square are called spacelike, timelike, or null
- 4-vectors also transform using Λ since: $\Lambda^{\mu}{}_{\nu} = \dfrac{\partial x'^{\mu}}{\partial x^{\nu}} \;\wedge\; \Lambda_{\nu}{}^{\mu} = \dfrac{\partial x^{\mu}}{\partial x'^{\nu}}$
- Contravariant 4-vectors: $A'^{\mu} = \Lambda^{\mu}{}_{\nu}\, A^{\nu}$ and covariant 4-vectors: $A'_{\mu} = \Lambda_{\mu}{}^{\nu}\, A_{\nu}$
- Tensor transformation rules (an example): $T'^{\alpha}{}_{\beta}{}^{\gamma} = A'^{\alpha}{}_{\beta} B'^{\gamma} = \Lambda^{\alpha}{}_{\delta} \Lambda_{\beta}{}^{\varepsilon} \Lambda^{\gamma}{}_{\zeta} T^{\delta}{}_{\varepsilon}{}^{\zeta}$
- In SR, massless particles (like photons) move on null paths and massive particles move on timelike paths, usually parametrized using proper time τ as $x^{\mu}(\tau)$
- Tangent vector in this parametrization is known as the **four-velocity** U^{μ} :
  $U^{\mu} = \dfrac{dx^{\mu}}{d\tau} = \gamma\left(c, v^1, v^2, v^3\right)$, where v^i are the components of particle's three-velocity $v = \sqrt{\left(v^1\right)^2 + \left(v^2\right)^2 + \left(v^3\right)^2}$, and γ is the Lorentz factor: $\gamma = 1 \Big/ \sqrt{1 - \dfrac{v^2}{c^2}}$
- In the rest frame of a particle its four-velocity is: $U^{\mu} = (c, 0, 0, 0)$

Energy-momentum tensor

- A complete description of the energy and momentum of an individual particle is provided by its energy-momentum four-vector: $p^\mu = mU^\mu = \left(E/c, p^1, p^2, p^3\right)$, where m is the rest mass and E is the relativistic energy of the particle: $E = \gamma\, mc^2$
- For an extended system comprised of huge number of particles, instead of specifying the individual momentum vectors of each particle, it is more convenient to describe the system as a **fluid** (a continuum of matter described by macroscopic quantities such as density, pressure, entropy, viscosity, etc.)
- A single momentum four-vector field is insufficient to describe the energy and momentum of a fluid and it is necessary to define the **energy-momentum tensor** $T^{\mu\nu}$ (also called **stress-energy tensor**)
- $T^{\mu\nu}$ is a symmetric (2, 0) tensor which describes all energy-like aspects of a system: energy density, pressure, stress, ...
- Formal definition: $T^{\mu\nu}$ is the flux of the μ-th component of the momentum vector p^μ across a surface with constant x^ν coordinate



- In GR, $T^{\mu\nu}$ is the source of the gravitational field and spacetime curvature
- All types of matter, from stars to the entire Universe, are modeled in GR as **perfect fluids** which can be completely characterized only by their **pressure** and **density**

Perfect fluid

- Perfect fluid is a fluid with no heat conduction and no viscosity, which looks **isotropic in thr rest frame**
- Due to isotropy in the rest frame, the energy-momentum tensor of prefect fluid $T^{\mu\nu}$ is diagonal, there is no net flux of any component of momentum in an orthogonal direction, and the nonzero spacelike components must all be equal: $T^{11} = T^{22} = T^{33}$
- The only two independent numbers are T^{00} (density) and one of the T^{ii} (pressure)
- Thus, $T^{\mu\nu}$ for perfect fluid in SR is given by: $T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu + p\,\eta^{\mu\nu}$, where ρ is rest frame **mass density**, p is **isotropic pressure** and U^μ is the 4-velocity of the fluid
- $T^{\mu\nu}$ takes on a particularly simple form in the rest frame: $T^{\mu\nu} = \begin{bmatrix} \rho_e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$, where $\rho_e = \rho c^2$ is the **energy density**
- The simplest example is **dust** (perfect fluid with zero pressure), defined as a collection of particles at rest with respect to each other, for which: $T^{\mu\nu}_{\text{dust}} = \rho_{\text{e}} U^\mu U^\nu$
- **Law of energy-momentum conservation in SR**: $T^{\mu\nu}$ in SR is being conserved since it has vanishing divergence: ${T^{\mu\nu}}_{,\mu} = \frac{\partial T^{\mu\nu}}{\partial x^\mu} = 0$
- This is a set of four equations (one for each value of v):
  - The $v = 0$ equation corresponds to conservation of energy
  - ${T^{\mu k}}_{,\mu} = 0$ expresses conservation of the kth component of the momentum

Exam questions

1. Reference frames, postulates of Special Relativity, Minkowski spacetime and Lorentz transformations
2. Tensor formulation of Special Relativity, 4-velocity, 4-momentum, energy-momentum tensor and perfect fluid

Literature

- Weinberg, S., 1972, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley-VCH
- Sean M. Carroll, 1997. *Lecture Notes on General Relativity*, arXiv, gr-qc/9712019

Exercise 1

- Find the matrix for the Lorentz transformation consisting of a boost v_x in then *x*-direction followed by a boost v_y in the *y*-direction. Show that the boosts performed in the reverse order would give a different transformation

Exercise 2

- Calculate the square $U \cdot U$ of the 4-velocity vector U^μ

Exercise 3

- The 4-velocity u^μ corresponds to 3-velocity v^i. Express:

  a) u^0 in terms of v

  b) u^i in terms of v

  c) u^0 in terms of u^i

  d) $d/d\tau$ in terms of d/dt and v

Exercise 4

- Express the rest mass of a particle in terms of the square of its 4-momentum

Exercise 5

- A particle of rest mass m_1 and velocity v collides with a stationary particle of rest mass m_2 and is absorbed by it. Find the rest mass m of the resultant compound system

Solution 1

$$\Lambda_x = \begin{bmatrix} \gamma_x & -\gamma_x v_x/c & 0 & 0 \\ -\gamma_x v_x/c & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Lambda_y = \begin{bmatrix} \gamma_y & 0 & -\gamma_y v_y/c & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_y v_y/c & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda_x \Lambda_y = \begin{bmatrix} \gamma_x \gamma_y & -\gamma_x v_x/c & -\gamma_x \gamma_y v_y/c & 0 \\ -\gamma_x \gamma_y v_x/c & \gamma_x & \gamma_x \gamma_y v_x v_y/c^2 & 0 \\ -\gamma_y v_y/c & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda_y \Lambda_x = \begin{bmatrix} \gamma_x \gamma_y & -\gamma_x \gamma_y v_x/c & -\gamma_y v_y/c & 0 \\ -\gamma_x v_x/c & \gamma_x & 0 & 0 \\ -\gamma_x \gamma_y v_y/c & \gamma_x \gamma_y v_x v_y/c^2 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \Lambda_x \Lambda_y \neq \Lambda_y \Lambda_x$$

Solution 2

$$U \cdot U = \eta_{\mu\nu} U^{\mu} U^{\nu} = -\left(\gamma\, c\right)^2 + \gamma^2\, v^2 = -c^2$$

Solution 3

a) $u^\mu = \left(u^0, u^1, u^2, u^3\right) = \gamma\left(c, v^1, v^2, v^3\right) \Rightarrow u^0 = \gamma\, c = \dfrac{c}{\sqrt{1-\dfrac{v^2}{c^2}}}$

b) $u^i = \gamma\, v^i = \dfrac{v^i}{\sqrt{1-\dfrac{v^2}{c^2}}}$

c) $u \cdot u = -\left(u^0\right)^2 + \left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2 = -c^2 \Rightarrow u^0 = \sqrt{c^2 + u^i\, u_i}$

d) $c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt^2\left(1-\dfrac{v^2}{c^2}\right) \Rightarrow \dfrac{dt}{d\tau} = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\cdot\frac{d}{dt} = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}\cdot\frac{d}{dt}$$

Solution 4

$$p^\mu = \left(\gamma\, mc, \gamma\, mv^1, \gamma\, mv^2, \gamma\, mv^3\right) \Rightarrow$$

$$p \cdot p = -\gamma^2 m^2 c^2 + \gamma^2 m^2 (v^1)^2 + \gamma^2 m^2 (v^2)^2 + \gamma^2 m^2 (v^3)^2$$

$$p \cdot p = m^2 \left(-\gamma^2 c^2 + \gamma^2 v^2\right) = -c^2 m^2 \quad \Rightarrow \quad m = \frac{1}{c}\sqrt{-p \cdot p}$$

Solution 5

- 4-momentum of the resultant system:

$$p^{\mu} = (\gamma\, m_1 c, \gamma\, m_1 v) + (m_2 c, 0)\,, \quad \gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \Rightarrow$$

$$p \cdot p = -\left(\gamma\, m_1 c + m_2 c\right)^2 + \left(\gamma\, m_1 v\right)^2 = -\gamma^2 m_1^2 c^2 - 2\gamma m_1 m_2 c^2 - m_2^2 c^2 + \gamma^2 m_1^2 v^2$$

$$p \cdot p = -m_1^2 c^2 - 2\gamma m_1 m_2 c^2 - m_2^2 c^2 \quad \wedge \quad m = \frac{1}{c}\sqrt{-p \cdot p} \quad \Rightarrow$$

$$m = \sqrt{m_1^2 + 2\gamma m_1 m_2 + m_2^2}$$