MASS 2023 Course:
Gravitation and Cosmology

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Lecture 06

- Covariant differentiation
  - Covariant gradient, curl, and divergence
- Parallel transport
- Minimal-coupling principle
  - GR expression for energy-momentum tensor of perfect fluid
- Riemann-Christoffel curvature tensor
- Ricci curvature tensor and scalar curvature
- Exercises

Covariant differentiation

- Ordinary differentiation of a tensor does not generally yield another tensor
- For instance, consider differentiating a contravariant vector V^μ with respect to $x^{\lambda'}$:

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} V^\nu \quad \Rightarrow \quad \frac{\partial V^{\mu'}}{\partial x^{\lambda'}} = \frac{\partial x^{\mu'}}{\partial x^\nu}\frac{\partial x^\rho}{\partial x^{\lambda'}}\frac{\partial V^\nu}{\partial x^\rho} + \frac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\rho}\frac{\partial x^\rho}{\partial x^{\lambda'}} V^\nu$$

- The first term is what would be expected if $\frac{\partial V^\mu}{\partial x^\lambda}$ was a tensor, but the second term destroys the tensor behavior
- However, it can be shown that: $\Gamma^{\mu'}_{\lambda'\kappa'} V^{\kappa'} = \frac{\partial x^{\mu'}}{\partial x^\nu}\frac{\partial x^\rho}{\partial x^{\lambda'}}\Gamma^\nu_{\rho\sigma}V^\sigma - \frac{\partial^2 x^{\mu'}}{\partial x^\rho \partial x^\sigma}\frac{\partial x^\rho}{\partial x^{\lambda'}}V^\sigma \Rightarrow$ the sum $\frac{\partial V^{\mu'}}{\partial x^{\lambda'}} + \Gamma^{\mu'}_{\lambda'\kappa'}V^{\kappa'} = \frac{\partial x^{\mu'}}{\partial x^\nu}\frac{\partial x^\rho}{\partial x^{\lambda'}}\left(\frac{\partial V^\nu}{\partial x^\rho} + \Gamma^\nu_{\rho\sigma}V^\sigma\right)$ is a tensor
- Therefore, a **covariant derivative** $V^\mu{}_{;\lambda}$ of a contravariant vector V^μ is defined as:

$$\boxed{V^\mu{}_{;\lambda} \equiv \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa}V^\kappa}$$

and it is a tensor because: $V^{\mu'}{}_{;\lambda'} = \frac{\partial x^{\mu'}}{\partial x^\nu}\frac{\partial x^\rho}{\partial x^{\lambda'}}V^\nu{}_{;\rho}$

- Analogously, a covariant derivative of a covariant vector V_μ can be defined as:

$$\boxed{V_{\mu;\nu} \equiv \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu}V_\lambda}$$

and it is also a tensor because: $V_{\mu';\nu'} = \frac{\partial x^\rho}{\partial x^{\mu'}}\frac{\partial x^\sigma}{\partial x^{\nu'}}V_{\rho;\sigma}$

- Covariant differentiation reduces to ordinary differentiation in the absence of gravitation, i.e. when $\Gamma^\mu_{\nu\lambda} = 0$

Covariant derivative of higher rank tensors

- Covariant derivative of a general tensor T of arbitrary rank with respect to x^σ is calculated by introducing a term with a single +Γ for each upper index, and a term with a single -Γ for each lower index:

$$\begin{aligned} T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l;\sigma} \quad = \quad & \frac{\partial T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l}}{\partial x^\sigma} \\ & +\Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \ldots \\ & -\Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\lambda\ldots\nu_l} - \ldots \end{aligned}$$

- **Important note**: commas (,) in lower indices are used for partial derivatives and semicolons (;) are used for covariant ones
- Alternative notation ∇_σ is often used for covariant derivative:

$$\nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \equiv T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l;\sigma}$$

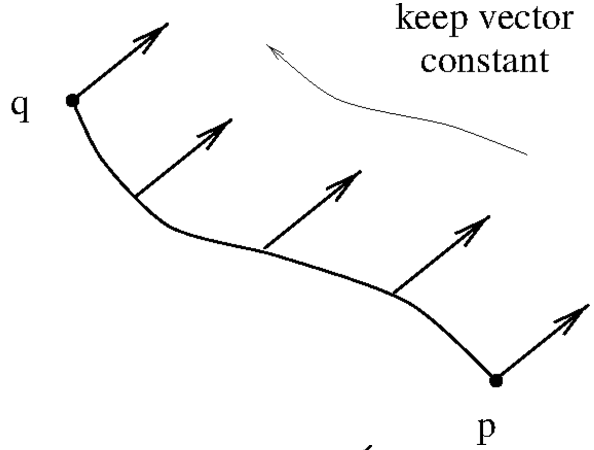
- Covariant derivative of the metric tensor is zero: $g_{\mu\nu;\lambda} = 0, \quad g^{\mu\nu}{}_{;\lambda} = 0, \quad g^{\mu}_{\nu;\lambda} = 0$
- Covariant differentiation and raising and lowering indices are commutative operations: $(g^{\mu\nu}V_\nu)_{;\lambda} = g^{\mu\nu}V_{\nu;\lambda} \quad \wedge \quad (g_{\mu\nu}V^\nu)_{;\lambda} = g_{\mu\nu}V^\nu{}_{;\lambda}$

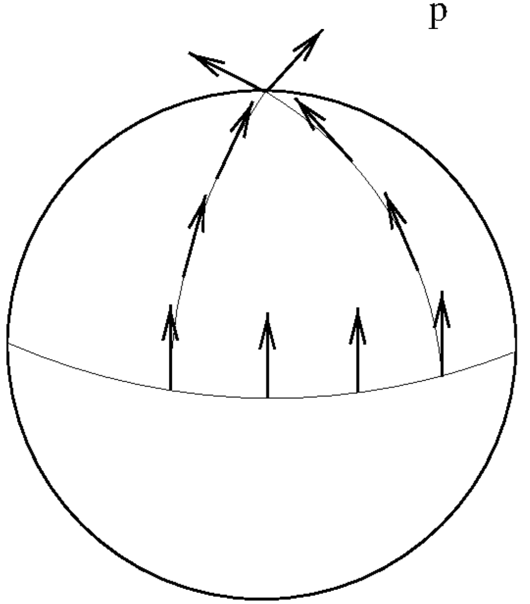
Covariant gradient, curl, and divergence

- Algebra of covariant differentiation:
  1. Covariant derivative of a linear combination of tensors is equal to linear combination of the covariant derivatives: $(\alpha A^{\mu}{}_{\nu} + \beta B^{\mu}{}_{\nu})_{;\lambda} = \alpha A^{\mu}{}_{\nu;\lambda} + \beta B^{\mu}{}_{\nu;\lambda}$
  2. Covariant derivative of a direct product of tensors obeys the **Leibniz rule**:
  $$\left(A^{\mu}{}_{\nu} B^{\lambda}\right)_{;\rho} = A^{\mu}{}_{\nu;\rho} B^{\lambda} + A^{\mu}{}_{\nu} B^{\lambda}{}_{;\rho}$$
  3. Covariant derivative of a contracted tensor is equal to contraction of the covariant derivative: $T^{\mu\lambda}{}_{\lambda;\rho} = \dfrac{\partial T^{\mu\lambda}{}_{\lambda}}{\partial x^{\rho}} + \Gamma^{\mu}_{\rho\nu} T^{\nu\lambda}{}_{\lambda}$
- Covariant derivative of a scalar S is ordinary **gradient**: $S_{;\mu} = \dfrac{\partial S}{\partial x^{\mu}}$
- **Covariant curl** of a covariant vector V_{μ} is ordinary curl: $V_{\mu;\nu} - V_{\nu;\mu} = \dfrac{\partial V_{\mu}}{\partial x^{\nu}} - \dfrac{\partial V_{\nu}}{\partial x^{\mu}}$
- **Covariant divergence** of a contravariant vector V^{μ}, a tensor $T^{\mu\nu}$, and an antisymmetric tensor $A^{\mu\nu} = -A^{\nu\mu}$: $V^{\mu}{}_{;\mu} = \dfrac{1}{\sqrt{|g|}} \dfrac{\partial\left(\sqrt{|g|}\, V^{\mu}\right)}{\partial x^{\mu}}$,
$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{|g|}} \frac{\partial\left(\sqrt{|g|}\, T^{\mu\nu}\right)}{\partial x^{\mu}} + \Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda}, \qquad A^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{|g|}} \frac{\partial\left(\sqrt{|g|}\, A^{\mu\nu}\right)}{\partial x^{\mu}}$$
- Cyclicity of covariant derivative of an antisymmetric covariant tensor $A_{\mu\nu} = -A_{\nu\mu}$:
$$A_{\mu\nu;\lambda} + A_{\lambda\mu;\nu} + A_{\nu\lambda;\mu} = \frac{\partial A_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial A_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial A_{\nu\lambda}}{\partial x^{\mu}}$$

Parallel transport of vectors

- In GR, vectors are elements of tangent spaces defined at individual points
- In order to do the usual operations with vectors (i.e. to add and subtract them, take their dot product, etc.) at different points in a vector space, it is necessary to move a vector from one point to another while keeping it constant
- **Parallel transport** is the concept of moving a vector along a path, keeping it constant all the while
- Parallel transport is defined whenever there is a connection
- In contrast to flat spaces, the result of parallel transporting a vector from one point to another in a curved space depends on the path taken between the points
- It is not possible to move a vector from one tangent space to another in an unique way
- Even the transport along a closed path does not preserve, in general, the direction of vectors
- Nevertheless, parallel transport can be defined so that it preserves the inner product of two vectors, their norm, orthogonality, etc
- If defined in that way, parallel transport can be considered as the curved-space generalization of the concept of keeping the vector constant as it is moved along a path





Example of parallel transport on two-sphere

Equation of parallel transport

- In order to define parallel transport of the tensor T along a curve, we first define **covariant derivative of the tensor T along the path** $x^\mu(\lambda)$ as:

$$\frac{D}{D\lambda}T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \equiv T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l;\sigma}\frac{dx^\sigma}{d\lambda}$$

- **Parallel transport of the tensor T along the path** $x^\mu(\lambda)$ is defined by requiring its covariant derivative along the path to vanish, i.e. by
- **Equation of parallel transport**: $\boxed{\frac{D}{D\lambda}T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} = 0}$
- Equation of parallel transport for a vector is: $\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0$
- Parallel transported tensor is a unique continuation of the tensor to other points along the path which solves the parallel transport equation
- $g_{\mu\nu}$ is always parallel transported: $\frac{D\,g_{\mu\nu}}{D\lambda} = g_{\mu\nu;\sigma}\frac{dx^\sigma}{d\lambda} = 0$
- In addition to being a path of the shortest distance between two points, geodesic is also a **path which parallel transports its own tangent vector** $\frac{dx^\mu}{d\lambda}$:

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 \quad \Rightarrow \quad \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

Minimal-coupling principle

- **Minimal-coupling principle** is the following simple recipe for generalizing laws of physics to the curved spacetime context:
  1. Take a law of physics, valid in inertial coordinates in flat spacetime
  2. Write it in a coordinate-invariant (tensorial) form
  3. Assert that the resulting law remains true in curved spacetime
- If $\eta_{\mu\nu}$ is replaced with $g_{\mu\nu}$ and all derivatives with covariant derivatives in an equation that holds in SR in the absence of gravitation, then the resulting equation will be generally covariant and true in the absence of gravitation, and therefore, according to the Principle of General Covariance, it will be true in the presence of gravitational field in a region of spacetime sufficiently small when compared with the scale of the gravitational field
- GR expressions for the energy-momentum tensor and the law of energy-momentum conservation could be obtained by generalizations of the corresponding SR expressions according to the minimal-coupling principle:
  1. GR expression for the energy-momentum tensor $T^{\mu\nu}$ of a perfect fluid:
  $$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu + p\, g^{\mu\nu}$$
  2. Energy-momentum tensor $T^{\mu\nu}$ in GR is being conserved since it has vanishing **covariant** divergence (**law of energy-momentum conservation in GR**):
  $$T^{\mu\nu}{}_{;\mu} = 0$$

Riemann-Christoffel curvature tensor

- Field equations that are generally covariant and that reduce to the proper form for weak fields can be obtained using tensors formed from $g_{\mu\nu}$ and its derivatives
- Since no new tensor can be constructed using only $g_{\mu\nu}$ and its first derivatives, it is necessary to include also the second derivatives of $g_{\mu\nu}$
- The only (unique) tensor that can be constructed from $g_{\mu\nu}$ and its first and second derivatives is the **Riemann** (or **Riemann-Christoffel**) **curvature tensor** $R^{\lambda}{}_{\mu\nu\kappa}$:

$$R^{\lambda}{}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}$$

- $R^{\lambda}{}_{\mu\nu\kappa}$ is is linear in the second derivatives of $g_{\mu\nu}$ and is the most common way used to express the **curvature of Riemannian manifolds**
- $R^{\lambda}{}_{\mu\nu\kappa}$ assigns a tensor to each point of a Riemannian manifold, i.e. it is a **tensor field**
- $R^{\lambda}{}_{\mu\nu\kappa}$ could be related to the Gaussian curvature K of a specially constructed 2D surface at a given point in a Riemannian space of an arbitrary number of dimensions
- The necessary and sufficient conditions for a metric $g_{\mu\nu}$ to be equivalent to the Minkowski metric $\eta_{\alpha\beta}$ (flat space) are that the curvature tensor calculated from $g_{\mu\nu}$ must everywhere vanish: $R^{\lambda}{}_{\mu\nu\kappa} = 0$
- In other words: if the components of the metric are constant in some coordinate system, the Riemann tensor will vanish, while if the Riemann tensor vanishes we can always construct a coordinate system in which the metric components are constant

Algebraic properties of Riemann tensor

- Algebraic properties of $R^{\lambda}{}_{\mu\nu\kappa}$ are more clear if its fully covariant form is considered:

$$R_{\lambda\mu\nu\kappa} \equiv g_{\lambda\sigma} R^{\sigma}{}_{\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right] + \\ + g_{\eta\sigma}\left[\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right]$$

- **Symmetries and identities** of $R_{\lambda\mu\nu\kappa}$:
  - Symmetry: $R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu}$
  - Antisymmetry: $R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = +R_{\mu\lambda\kappa\nu}$
  - Cyclicity: $R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0$
- Due to this, there are $n^2(n^2-1)/12$ independent components of the Riemann tensor in n dimensions $\Rightarrow$ only 20 independent components of $R_{\lambda\mu\nu\kappa}$ in 4 dimensions
- Covariant derivative of the $R_{\lambda\mu\nu\kappa}$:

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2}\frac{\partial}{\partial x^\eta}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right)$$

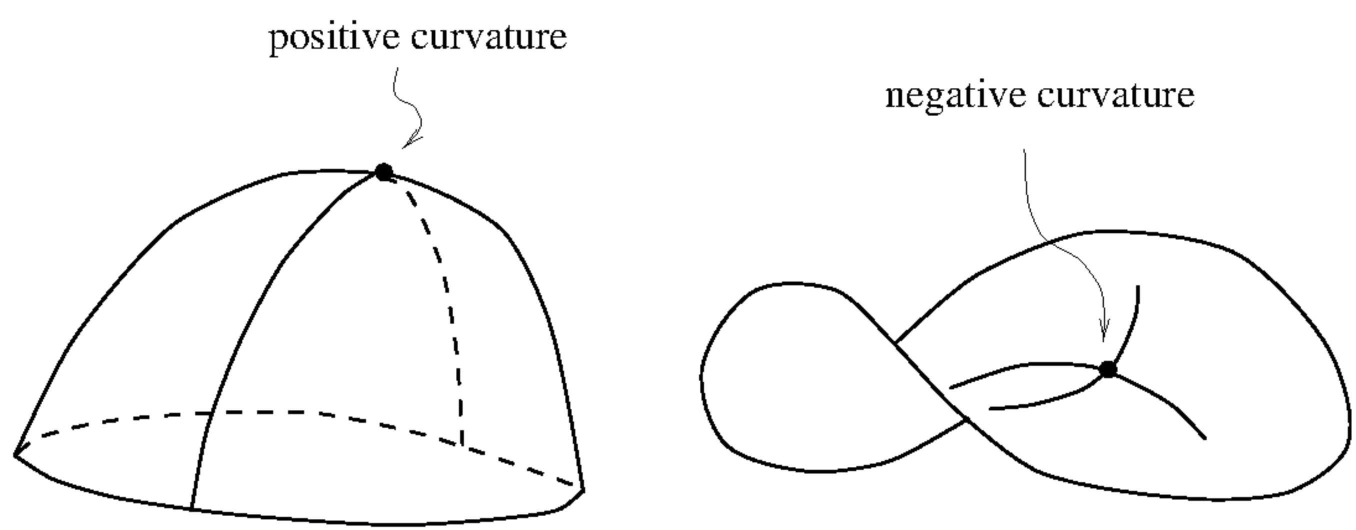
- **Bianchi identities** are differential identities obtained by cyclically permuting the indices ν, κ and η in covariant derivative $R_{\lambda\mu\nu\kappa;\eta}$: $R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0$

Ricci curvature tensor and scalar curvature

- For the curvature tensor formed from an arbitrary (not necessarily Christoffel) connection, there are a number of independent contractions to take
- In the case of the Christoffel connection, the only independent is a contraction known as **Ricci tensor**: $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$
- In terms of affine connection $R_{\mu\nu}$ is:

$$R_{\mu\kappa} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\eta}$$

- As a consequence of the various symmetries of Riemann tensor, Ricci tensor is symmetric: $R_{\mu\nu} = R_{\nu\mu}$
- **Ricci scalar** is formed by metric contraction of Ricci tensor: $R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$
- **Einstein tensor**: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$ which has vanishing divergence: $G^{\mu}{}_{\nu;\mu} = 0$ (implied by Bianchi identities)
- Ricci scalar is constant for a given space
- Spaces are **positively curved** if they have a positive Ricci scalar, and **negatively curved** (saddle-like) if they have a negative Ricci scalar



Exam questions

1. Covariant differentiation, parallel transport and minimal-coupling principle
2. Riemann-Christoffel curvature tensor, Ricci curvature tensor and Ricci scalar curvature

Literature

- Weinberg, S., 1972, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley-VCH
- Sean M. Carroll, 1997. *Lecture Notes on General Relativity*, arXiv, gr-qc/9712019

Exercise 1

- Evaluate the following covariant derivatives: $T_{\mu\nu;\sigma}$, $T^{\mu\nu}{}_{;\sigma}$, $T^{\mu}{}_{\nu;\sigma}$, $T^{\mu\sigma}{}_{\lambda;\rho}$

Exercise 2

- Prove that covariant derivative of the Kronecker delta symbol $\delta^{\mu}{}_{\nu}$ is equal to zero

Exercise 3

- Use Leibniz rule to derive the expression for covariant derivative of a second rank tensor $T_{\mu\nu}$, knowing that it could be written as a direct product of two covariant vectors A_μ and B_ν: $T_{\mu\nu} = A_\mu B_\nu$.

Exercise 4

- For the two-sphere metric $ds^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$, calculate:
  a) all nonzero components of the Riemann tensor
  b) all nonzero components of the Ricci tensor
  c) value of Ricci scalar

Exercise 5

- Find the Christoffel symbols and Riemann curvature components for the two dimensional spacetime: $ds^2 = dv^2 - v^2 du^2$

Solution 1

$$T_{\mu\nu;\sigma} = \frac{\partial T_{\mu\nu}}{\partial x^\sigma} - \Gamma^\lambda_{\sigma\mu} T_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} T_{\mu\lambda}$$

$$T^{\mu\nu}{}_{;\sigma} = \frac{\partial T^{\mu\nu}}{\partial x^\sigma} + \Gamma^\mu_{\sigma\lambda} T^{\lambda\nu} + \Gamma^\nu_{\sigma\lambda} T^{\mu\lambda}$$

$$T^\mu{}_{\nu;\sigma} = \frac{\partial T^\mu{}_\nu}{\partial x^\sigma} + \Gamma^\mu_{\sigma\lambda} T^\lambda{}_\nu - \Gamma^\lambda_{\sigma\nu} T^\mu{}_\lambda$$

$$T^{\mu\sigma}{}_{\lambda;\rho} = \frac{\partial T^{\mu\sigma}{}_\lambda}{\partial x^\rho} + \Gamma^\mu_{\rho\nu} T^{\nu\sigma}{}_\lambda + \Gamma^\sigma_{\rho\nu} T^{\mu\nu}{}_\lambda - \Gamma^\kappa_{\lambda\rho} T^{\mu\sigma}{}_\kappa$$

Solution 2

- Kronecker delta symbol is a second rank tensor, so:

$$T^{\mu}{}_{\nu;\sigma} = \frac{\partial T^{\mu}{}_{\nu}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\sigma\lambda} T^{\lambda}{}_{\nu} - \Gamma^{\lambda}_{\sigma\nu} T^{\mu}{}_{\lambda} \Rightarrow$$

$$\delta^{\mu}{}_{\nu;\sigma} = \frac{\partial \delta^{\mu}{}_{\nu}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\sigma\lambda} \delta^{\lambda}_{\nu} - \Gamma^{\lambda}_{\sigma\nu} \delta^{\mu}_{\lambda} = 0 + \Gamma^{\mu}_{\sigma\nu} - \Gamma^{\mu}_{\sigma\nu} = 0$$

Solution 3

$$T_{\mu\nu} = A_\mu B_\nu \Rightarrow T_{\mu\nu;\sigma} = \left(A_\mu B_\nu\right)_{;\sigma} = A_{\mu;\sigma} B_\nu + A_\mu B_{\nu;\sigma} =$$

$$= \left(\frac{\partial A_\mu}{\partial x^\sigma} - \Gamma^\lambda_{\mu\sigma} A_\lambda\right) B_\nu + \left(\frac{\partial B_\nu}{\partial x^\sigma} - \Gamma^\lambda_{\nu\sigma} B_\lambda\right) A_\mu =$$

$$= \frac{\partial \left(A_\mu B_\nu\right)}{\partial x^\sigma} - \Gamma^\lambda_{\mu\sigma} A_\lambda B_\nu - \Gamma^\lambda_{\nu\sigma} A_\mu B_\lambda =$$

$$= \frac{\partial T_{\mu\nu}}{\partial x^\sigma} - \Gamma^\lambda_{\sigma\mu} T_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} T_{\mu\lambda}$$

Solution 4

a) Nonzero components of metric tensor and inverse metric tensor:

$$g_{\theta\theta} = a^2, \quad g_{\phi\phi} = a^2 \sin^2\theta \quad \wedge \quad g^{\theta\theta} = a^{-2}, \quad g^{\phi\phi} = a^{-2}\sin^{-2}\theta$$

- Nonzero Christoffel symbols: $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$
- Nonzero components of the Riemann tensor:

$$R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi} = a^2\sin^2\theta$$

a) Nonzero components of the Ricci tensor: $R_{\theta\theta} = 1, \quad R_{\phi\phi} = \sin^2\theta$

b) Ricci scalar: $R \equiv R^{\mu}_{\mu} = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \dfrac{2}{a^2}$

Note that Ricci scalar *R* is proportional to Gaussian curvature *K*, since Gaussian curvature of sphere with radius *a* is $K = \dfrac{1}{a^2}$

Solution 5

The only nonvanishing Christoffel symbols are $\Gamma^v_{uu} = v$ and $\Gamma^u_{uv} = \Gamma^u_{vu} = v^{-1}$ $\Rightarrow$

$$R_{vuvu} = R^v{}_{uvu} = \Gamma^v_{uu,v} - \Gamma^v_{vu,u} + \Gamma^v_{v\alpha}\Gamma^\alpha_{uu} - \Gamma^v_{u\alpha}\Gamma^\alpha_{uv} = 1 - 0 + 0 - 1 = 0$$

Since there is only one independent Riemann component in two dimensions we conclude that $R_{\alpha\beta\gamma\delta} = 0,$ and the spacetime is flat