MASS 2023 Course:
Gravitation and Cosmology

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Lecture 07

- Calculus of variations
  - Functional and its variation
  - Extremals
  - Euler-Lagrange equation
- Principle of least action (Hamilton's principle)
  - Lagrangian and Lagrangian density
  - Action
- Einstein field equations
  - Hilbert action
  - Action in the presence of matter and cosmological constant
  - Varying the action with respect to the metric
- Exercises

Calculus of variations

- **Functional** $J[y]$ is a correspondence J: $y(x) \to R$ which assigns a definite (real) number to each function (or curve) $y = y(x)$ belonging to some class
- Functional is a kind of function, where the independent variable is itself a function
- Example: arc length of a curve $y = y(x)$ in Euclidean plane: $J[y] = \int_a^b \sqrt{1+y'^2}\,dx$
- More general functionals of the following form are of particular importance:

$$J[y] = \int_a^b F[x, y(x), y'(x)]\,dx, \quad y(a) = A, \quad y(b) = B,$$

  where $F(x, y, z)$ is a continuous function of three variables, and $y(x)$ is a continuously differentiable function defined on the interval $[a, b]$
- By choosing different functions $F(x, y, z)$ we obtain different functionals: e.g. if $F(x,y,z) = \sqrt{1+z^2}$, then $J[y]$ is the arc length of the curve $y = y(x)$
- **Increment** ΔJ of the functional $J[y]$ corresponding to the increment $h = h(x)$ of the "independent variable" $y = y(x)$ is:

$$\Delta J = J[y+h] - J[y] = \int_a^b \left[F(x, y+h, y'+h') - F(x, y, y')\right] dx$$

- Taylor expansion $\Rightarrow \Delta J = \int_a^b \left[F_y(x, y, y')\,h + F_{y'}(x, y, y')\,h'\right] dx + \ldots$

Euler-Lagrange differential equation

- **Variation δJ of functional** $J[y]$ is the first term in the previous expansion, i.e. the linear part of the increment ΔJ:

$$\delta J = \int_a^b \left[F_y\left(x,y,y'\right)h + F_{y'}\left(x,y,y'\right)h'\right]dx = \int_a^b \left(\frac{\partial F}{\partial y}h + \frac{\partial F}{\partial y'}h'\right)dx$$

- **Calculus of variations** is used for finding the maxima and minima of functionals
- **Extremal** is the function $y = y(x)$ for which functional $J[y]$ has an extremum
- A necessary condition for $J[y]$ to have an extremum for $y = y(x)$ is:

$$\delta J = 0 \quad \Rightarrow \quad \boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0}$$ **(Euler-Lagrange differential equation)**

- Extremals are obtained by solving the Euler-Lagrange differential equation
- Solution of this second-order differential equation will depend on two arbitrary constants, which are determined from the boundary conditions: $y(a) = A$ and $y(b) = B$
- Special cases where Euler-Lagrange equation can be reduced to a first-order differential equation, or where its solution can be obtained by evaluating integrals:

1. F does not depend on y: $F = F(x,y') \;\Rightarrow\; \dfrac{\partial F}{\partial y'} = C \;\Rightarrow\; y' = f(x,C)$

2. F does not depend on x: $F = F(y,y') \;\Rightarrow\;$ The first integral is: $F - y'\dfrac{\partial F}{\partial y'} = C$

3. F does not depend on y': $F = F(x,y) \;\Rightarrow\; \dfrac{\partial F}{\partial y} = 0$ (is not a differential equation)

Variational derivative

- **Variational (or functional) derivative** relates a change in a functional to a change in a function on which the functional depends
- In the case of functionals of the type: $J[y] = \int_a^b F\left[x, f(x), f'(x)\right]\, dx$, variational derivative is the left-hand side of Euler-Lagrange equation: $\dfrac{\delta J}{\delta f} = \dfrac{\partial F}{\partial f} - \dfrac{d}{dx}\dfrac{\partial F}{\partial f'}$
- Functional $J[y]$ has an extremum if its variational derivative vanish at every point, like in the case of a function which has an extremum if all of its partial derivatives vanish
- The analogs of all the familiar rules obeyed by ordinary derivatives (e.g. the formulas for differentiating sums and products of functions, composite functions, etc.) are also valid for variational derivatives:
  - Linearity: $\dfrac{\delta(\lambda F + \mu G)[\rho]}{\delta\rho(x)} = \lambda\dfrac{\delta F[\rho]}{\delta\rho(x)} + \mu\dfrac{\delta G[\rho]}{\delta\rho(x)}$, where λ and μ are constants
  - Product rule: $\dfrac{\delta(FG)[\rho]}{\delta\rho(x)} = \dfrac{\delta F[\rho]}{\delta\rho(x)}G[\rho] + F[\rho]\dfrac{\delta G[\rho]}{\delta\rho(x)}$
  - Chain rules: $\dfrac{\delta F[G[\rho]]}{\delta\rho(y)} = \int dx \left.\dfrac{\delta F[G]}{\delta G(x)}\right|_{G=G[\rho]} \cdot \dfrac{\delta G[\rho](x)}{\delta\rho(y)}$

$$\frac{\delta F[g(\rho)]}{\delta\rho(y)} = \frac{\delta F[g(\rho)]}{\delta g[\rho(y)]}\,\frac{dg(\rho)}{d\rho(y)}$$

- Functional derivative with respect to the metric is used for variation of the Hilbert action in order to obtain the Einstein field equations

Lagrangian and action

- Application of calculus of variations to classical mechanics
- Kinetic energy T of a system of n particles with masses m_i and coordinates x_i, y_i, z_i ($i = 1,\ldots, n$), where no constraints whatsoever are imposed on the system, is:

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)$$

- Potential energy U of the system is a function $U = U(t, x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$ such that the force acting on the ith particle has components:

$$X_i = -\frac{\partial U}{\partial x_i},\ Y_i = -\frac{\partial U}{\partial y_i},\ Z_i = -\frac{\partial U}{\partial z_i}$$

- **Lagrangian** $L = T - U$ of the system of particles is a function of the time t, positions (x_i, y_i, z_i) and velocities $(\dot{x}_i, \dot{y}_i, \dot{z}_i)$ of the n particles in the system
- **Action** is the functional given by the integral of Lagrangian: $S = \int_{t_0}^{t_1} L\, dt$
- **Principle of least (stationary) action or Hamilton's principle**: The motion of a system of n particles during the time interval $[t_0, t_1]$ is described by those functions $x_i(t)$, $y_i(t)$, $z_i(t)$, $i = 1,\ldots,n$, for which the action S is a minimum
- $\delta S = 0$ and the Euler-Lagrange equations must be satisfied for $i = 1,\ldots,$ n:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0,$$

$$\frac{\partial L}{\partial y_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_i} = 0,$$

$$\frac{\partial L}{\partial z_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}_i} = 0$$

Principle of least (stationary) action

- In classical mechanics, the parameters that define the configuration of a system are called **generalized coordinates** $q(t)$, and the space defined by these coordinates is called the **configuration space** of the physical system
- **Action** in generalized coordinates: $S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$
- **Principle of least action**: the path taken by the system between times t_1 and t_2 and configurations q_1 and q_2 is the one for which the action is stationary ($\delta S = 0$)
- **Euler-Lagrange equation**: $\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial (\dot{q})}\right) = 0$
- **Action**:
  - describes how a physical system has changed over time
  - is a functional which takes the trajectory (or path) of the system as its argument and has a real number as its result, taking different values for different paths
  - has dimensions of energy · time: [S] = J · s
- **Principle of least (stationary) action**:
  - is a variational principle that, when applied to the action of a mechanical system, yields the equations of motion for that system
  - is a most fundamental principle in classical mechanics, electromagnetism, general relativity, quantum mechanics, particle physics, ...

Hilbert action

- In the field theory, Lagrangian as a function of generalized coordinates $q(t)$ is replaced by a **Lagrangian density** $\mathcal{L}$, a function of the fields φ_i in the system, their derivatives and the spacetime coordinates themselves: $\mathcal{L} = \mathcal{L}\left(\varphi_i, \partial\varphi_i/\partial x_\mu, x_\mu\right)$
- Lagrangian is the spatial volume integral of the Lagrangian density: $L = \int \mathcal{L}\, d^3x$
- Often, a "Lagrangian density" is simply referred to as a "Lagrangian"
- Action S is then given by: $S = \int dt\, L = \int d^4x\, \mathcal{L}\left(\varphi_i, \partial\varphi_i/\partial x_\mu, x_\mu\right)$
- The equations of motion are obtained using **action principle**, written as: $\dfrac{\delta S}{\delta \varphi_i} = 0$
- Field equations of GR were first derived by Hilbert using the action principle, taking into account that in GR:
  - metric $g_{\mu\nu}$ is the field variable
  - invariant volume element is the scalar $\sqrt{-g}\, d^4x$, where $g = \det g_{\mu\nu}$
- Hilbert figured that the simplest possible choice for Lagrangian density is the one which is proportional to Ricci scalar R: $\mathcal{L}_H = \dfrac{R}{2\kappa}$, where $\kappa = \dfrac{8\pi G}{c^4}$ is the **Einstein gravitational constant**
- **Hilbert action** is then given by: $S_H = \int \sqrt{-g}\, d^4x \mathcal{L}_H = \dfrac{1}{2\kappa} \int R\sqrt{-g}\, d^4x$

Action in the presence of matter and cosmological constant

- Hilbert action S_H represents the gravitational part of the full action S, and its variation with respect to the metric leads to the vacuum field equations of GR
- Einstein equations in the presence of matter are derived using the full Lagrangian density $\mathcal{L}$, which is obtained by adding a term $\mathcal{L}_M$ describing matter fields (non-gravitational part of $\mathcal{L}$) to the Hilbert term: $\mathcal{L} = \mathcal{L}_H + \mathcal{L}_M$
- The full action is then: $S = \int \left[\frac{1}{2\kappa} R + \mathcal{L}_{\mathrm{M}} \right] \sqrt{-g}\, d^4x,$
- In that case the energy-momentum tensor is defined as: $T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta \left(\mathcal{L}_M \sqrt{-g} \right)}{\delta g^{\mu\nu}}$
- If cosmological constant Λ is included in the Lagrangian: $\mathcal{L} = \frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_M,$

  and then the full action is: $S = \int \left[\frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_M \right] \sqrt{-g}\, d^4x$
- Variation of the above action with respect to the metric leads to the Einstein field equations with cosmological constant Λ

Einstein field equations (EFE)

- According to the principle of least action, the variation of the action with respect to the inverse metric is zero: $\delta S = \delta \left\{ \int \left[\frac{1}{2\kappa} R + \mathcal{L}_{\mathrm{M}} \right] \sqrt{-g}\, d^4x \right\} = 0 \quad \Leftrightarrow$

$$\int \left[\frac{1}{2\kappa} \frac{\delta\left(\sqrt{-g}R\right)}{\delta g^{\mu\nu}} + \frac{\delta\left(\sqrt{-g}\mathcal{L}_M\right)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu}\, d^4x = 0 \quad \Leftrightarrow$$

$$\int \left[\frac{1}{2\kappa} \left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_M\right)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \sqrt{-g}\, d^4x = 0 \quad \Rightarrow$$

$$\frac{1}{2\kappa} \left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_M\right)}{\delta g^{\mu\nu}} = 0 \quad \Rightarrow$$

$$\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \frac{-2\kappa}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \Leftrightarrow \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \kappa\, T_{\mu\nu} \quad (1)$$

- Variation of the Ricci scalar: $\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \quad (2)$
- Variation of the metric determinant: $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,(g_{\mu\nu}\delta g^{\mu\nu}) \quad (3)$
- **Einstein field equations**: $(1) \wedge (2) \wedge (3) \quad \Rightarrow \quad \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa\, T_{\mu\nu}}$
- In the case with cosmological constant Λ: $\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa\, T_{\mu\nu}}$
- EFE relate the geometry of spacetime to the distribution of matter within it
- Wheeler: "Matter tells spacetime how to curve. Spacetime tells matter how to move."

Exam questions

1. Euler-Lagrange equation, Lagrangian and principle of least action
2. Hilbert action, action in the presence of matter and cosmological constant, Einstein field equations

Literature

- I. M. Gelfand and S. V. Fomin, 1963, *Calculus of Variations*, Revised English Edition, Prentice-Hall, Inc
- Weinberg, S., 1972, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley-VCH
- Sean M. Carroll, 1997. *Lecture Notes on General Relativity*, arXiv, gr-qc/9712019

Exercise 1

- Among all the curves $y = f(x)$ in the Euclidean plane, find the one which has the shortest arc length $A[y] = \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2}\, dx$ between two given points (x_1, y_1) and (x_2, y_2).

Exercise 2

- Find the extremal of the following functional:

$$J[y] = \int_1^2 \frac{\sqrt{1 + y'^2}}{x} dx, \quad y(1) = 0, \ y(2) = 1$$

Exercise 3

- For the 2-dimensional metric $ds^2 = (dx^2 - dt^2)/t^2$, find all timelike geodesic curves using the principle of least action.

Solution 1

- In order to find the extremal *f*(*x*) that minimizes the functional *A*[*y*] we have to solve the Euler-Lagrange equation: $\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0$, where: $L = \sqrt{1 + [f'(x)]^2}$
- Since *f* does not appear explicitly in *L*, the first term in the Euler–Lagrange equation vanishes and thus: $\frac{d}{dx}\frac{\partial L}{\partial f'} = 0 \quad \Rightarrow \quad \frac{d}{dx}\frac{f'(x)}{\sqrt{1+[f'(x)]^2}} = 0 \quad \Rightarrow$

$$\frac{f'(x)}{\sqrt{1+[f'(x)]^2}} = c \quad \Rightarrow \quad f'(x) = \frac{c}{\sqrt{1-c^2}} = A \quad \Rightarrow \quad f(x) = Ax + B$$

- Constants A and B are obtained from the boundary conditions:

$$A = \frac{y_2 - y_1}{x_2 - x_1} \wedge B = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

- The extremal *f*(*x*) that minimizes the functional *A*[*y*] is a straight line.

Solution 2

$L = \frac{\sqrt{1+y'^2}}{x}$ does not contain *y*, so Euler-Lagrange equation has the form:

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial y'} = C \quad \Rightarrow \quad \frac{y'}{x\sqrt{1+y'^2}} = C \quad \Leftrightarrow$$

$$y'^2\left(1 - C^2x^2\right) = C^2x^2 \quad \Leftrightarrow \quad y' = \frac{Cx}{\sqrt{1-C^2x^2}} \quad \Rightarrow \quad y = \frac{\sqrt{1-C^2x^2}}{C} + C_1$$

$$(y - C_1)^2 + x^2 = \frac{1}{C^2}$$

- From the boundary conditions, we find that: $C = \frac{1}{\sqrt{5}},\ C_1 = 2$,and the final solution is: $(y-2)^2 + x^2 = 5$

Solution 3

- Let a geodesic be $x(t)$. Then the action is: $S = \int ds = \int \sqrt{\dot{x}^2 - 1}\,\frac{dt}{t}$ and $\delta S = \delta \int \sqrt{\dot{x}^2 - 1}\,\frac{dt}{t} = 0$
- In this case Euler-Lagrange equation is: $\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$ and $L = \frac{\sqrt{\dot{x}^2 - 1}}{t} \Rightarrow$

$$\frac{d}{dt}\left(\frac{\dot{x}}{t\sqrt{\dot{x}^2 - 1}}\right) = 0 \Leftrightarrow \frac{\dot{x}}{t\sqrt{\dot{x}^2 - 1}} = c \Rightarrow \dot{x} = \pm\frac{c\,t}{\sqrt{c^2t^2 - 1}} \Rightarrow$$

$$x - x_0 = \pm\sqrt{t^2 - c^{-2}} \Rightarrow t^2 - (x - x_0)^2 = \frac{1}{c^2} \Rightarrow \boxed{\frac{t^2}{(1/c)^2} - \frac{(x - x_0)^2}{(1/c)^2} = 1}$$

- Geodesics are hyperbolas asymptotic to the light cones: