MASS 2023 Course:
Gravitation and Cosmology

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Lecture 08

- Vacuum solutions to the field equations of GR
  - Schwarzschild metric
  - Kerr metric
  - Reissner-Nordström metric
  - Kerr-Newman metric
- Black holes
- Classic Solar System tests of GR:
  - Perihelion precession of Mercury's orbit
  - Deflection of light by the Sun
  - Gravitational redshift of light
  - Gravitational (Shapiro) time delay
- Other experimental tests of GR:
  - Precession of orbiting gyroscopes (Lense-Thirring effect)
  - Discovery of the central supermassive black hole (SMBH) of the Milky Way
  - Observed images of the SMBH shadows at the centers of M87 and our Galaxy
  - Relativistically broadened spectral lines from Active Galactic Nuclei
  - Gravitational waves and their detection
- Exercises

Schwarzschild metric

- Einstein field equations (EFE) are a system of 10 partial differential equations in which the metric tensor $g_{\mu\nu}$ can be solved for
- **Spacetime metrics** $\mathbf{g}_{\mu\nu}$ are **solutions to EFE**, which can be exact or non-exact
- **Vacuum solutions** to EFE are those for which $T_{\mu\nu} = 0,$ and they describe regions in which no matter or non-gravitational fields are present
- **Schwarzschild metric**, found by Karl Schwarzschild in 1916, is an exact vacuum solution to EFE that describes the gravitational field outside a **spherically symmetric**, **non-rotating and uncharged** mass M

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) c^2 dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

where $\boxed{R_S = \frac{2GM}{c^2}}$ is the **Schwarzschild radius**, and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric on a unit two-sphere

- Any object whose radius is smaller than its R_S is called a **black hole**
- In the case of a non-rotating black hole, the surface at R_S acts as an **event horizon** (boundary beyond which events cannot affect an observer)
- R_S of Sun is ~3 km, R_S of Earth is ~9 mm and R_S of Moon is ~0.1 mm
- Since $g_{00} = -\left(1 - \frac{2GM}{c^2 r}\right)$, Schwarzschild metric should reduce to the weak field case when $r \gg R_S$

Kerr metric

- **Kerr metric** is a generalization of the Schwarzschild metric to a rotating mass
- It is an exact vacuum solution to EFE, found by Roy Kerr in 1963, that describes the gravitational field around a **rotating**, **uncharged**, and **axially symmetric** mass M

$$ds^2 = -\left(1 - \frac{R_S r}{\Sigma}\right) c^2 dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{R_S r a^2}{\Sigma} \sin^2\theta\right) \sin^2\theta \, d\phi^2$$

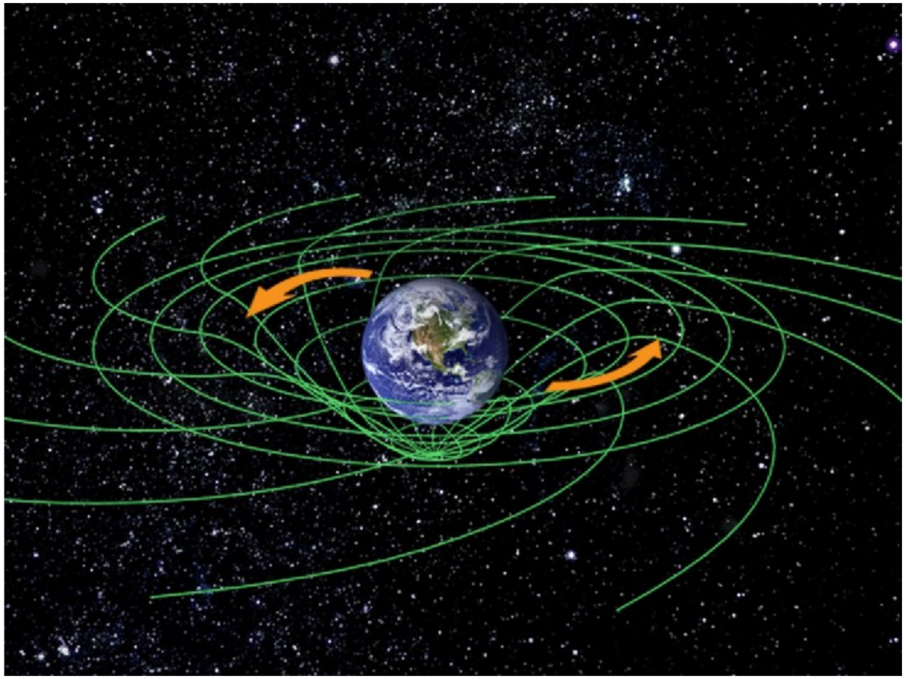
$$- \frac{2 R_S r a \sin^2\theta}{\Sigma} c dt \, d\phi, \text{ where } \Sigma = r^2 + a^2 \cos^2\theta, \quad \Delta = r^2 - R_S r + a^2$$

- $a = \dfrac{J}{Mc}$ is the **spin** (angular momentum J normalized to mass M)
- **Event horizon** of a rotating black hole (place where the purely radial component g_{rr} of the metric goes to infinity): $R_H^{\pm} = \dfrac{R_S \pm \sqrt{R_S^2 - 4a^2}}{2}$
- **Ergosphere** (place where the purely temporal component g_{tt} of the metric changes sign from positive to negative):

$$R_E^{\pm} = \frac{R_S \pm \sqrt{R_S^2 - 4a^2 \cos^2\theta}}{2}$$

- **Frame-dragging**: in Kerr metric, the reference frame is pulled by the rotating central mass to co-rotate with it, having angular speed:

$$\Omega = \frac{R_S r a c}{\Sigma \left(r^2 + a^2\right) + R_S r a^2 \sin^2\theta}$$



Reissner-Nordström metric

- **Reissner–Nordström metric** is an exact solution to EFE, that describes the gravitational field around a charged, non-rotating and spherically symmetric mass M

$$ds^2 = \left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right) c^2\, dt^2 - \left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2,$$

where R_Q is a characteristic length scale given by $R_Q^2 = \frac{Q^2 G}{4\pi\varepsilon_0 c^4}$, that corresponds to the charge Q, and ε_0 is the vacuum permittivity (electric constant)

- Two concentric event horizons: $R_H^{\pm} = \frac{1}{2}\left(R_S \pm \sqrt{r_S^2 - 4R_Q^2}\right)$

Kerr-Newman metric

- **Kerr-Newman metric** is the most general asymptotically flat exact solution to EFE that describes the gravitational field around a charged and rotating mass M

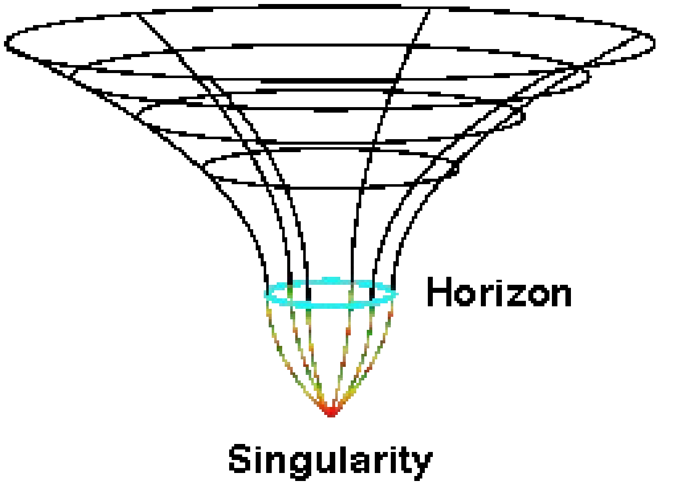
$$ds^2 = -\left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 + \left(c\, dt - a\sin^2\theta\, d\phi\right)^2 \frac{\Delta}{\rho^2} - \left(\left(r^2 + a^2\right) d\phi - ac\, dt\right)^2 \frac{\sin^2\theta}{\rho^2},$$

where $\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - R_S r + a^2 + R_Q^2$

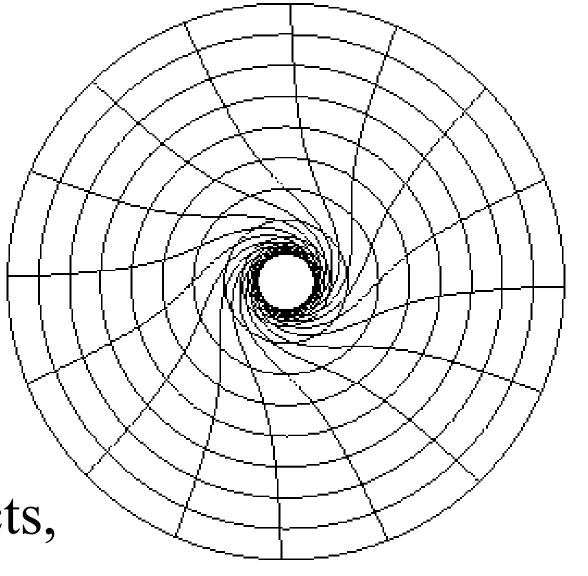
- Inner and outer event horizon: $R_H^{\pm} = \frac{R_S}{2} \pm \sqrt{\frac{R_S^2}{4} - a^2 - R_Q^2}$
- Inner and outer ergosphere: $R_E^{\pm} = \frac{R_S}{2} \pm \sqrt{\frac{R_S^2}{4} - a^2\cos^2\theta - R_Q^2}$

Black holes

- **Black hole** is a region of spacetime where gravity is so strong that nothing, not even light, can escape its event horizon
- Only three measurable parameters (not including the Hawking temperature): **mass**, angular momentum (**spin**) and **charge**
- Classification according to the **metric**:
  1. Schwarzschild (non-rotating and uncharged)
  2. Kerr (rotating and uncharged)
  3. Reissner–Nordström (non-rotating and charged)
  4. Kerr–Newman (rotating and charged)



- Classification according to their **masses**:
  1. Mini, micro or quantum mechanical: $M_{BH} << M_{\odot}$
  (primordial black holes in the early universe)
  2. Stellar-mass: $M_{BH} < 10^2\ M_{\odot}$
  (in the X-ray binary systems)
  3. Intermediate-mass: $M_{BH} \sim 10^2 - 10^5\ M_{\odot}$
  (in the centers of globular clusters)
  4. Supermassive: $M_{BH} \sim 10^5 - 10^{10}\ M_{\odot}$
  (in the centers of all galaxies, including ours)



- Astrophysical black holes are gravitationally collapsed objects, and they are usually assumed to be electrically neutral

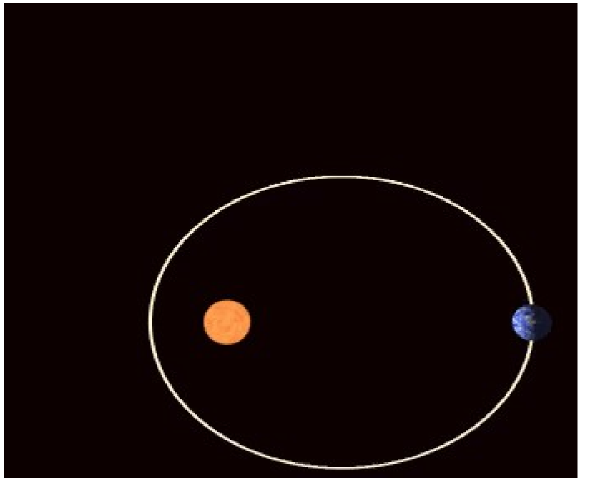
Experimental tests of GR

- Four classic Solar System tests of GR:
  - Perihelion precession of Mercury's orbit
  - Deflection of light by the Sun
  - Gravitational redshift of light
  - Gravitational (Shapiro) time delay
- Other experimental tests of GR:
  - Precession of orbiting gyroscopes (Lense-Thirring effect)
  - Discovery of the central SMBH of the Milky Way
  - Observed images of the SMBH shadows at the centers of M87 and our Galaxy
  - Relativistically broadened spectral lines from Active Galactic Nuclei (AGN)
  - Gravitational waves and their detection
- Because of these successes, **GR is** considered as modern and **standard theory of gravity**

Bound orbits: precession of perihelia

- In GR, a freely falling particle or photon can move along a bound or unbound orbit (geodesic) in the gravitational field of a central mass M

- Schwarzschild precession of elliptical orbits:



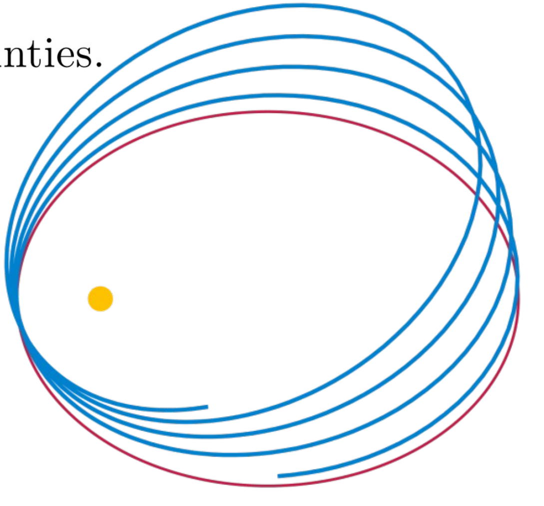
$$\Delta\varphi = \frac{6\pi GM}{c^2\, a\,(1-e^2)}$$

- GR prediction for perihelion precession is confirmed with high accuracy by recent high-precision observations of the Solar System bodies and spacecrafts

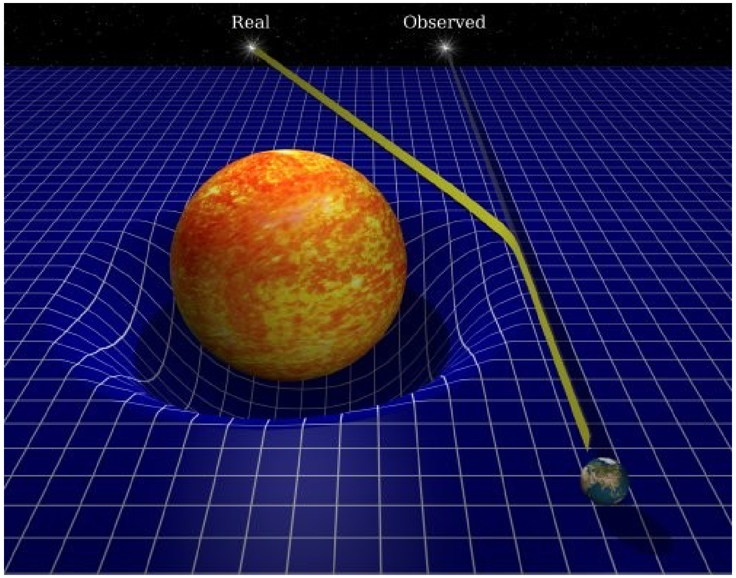
- In 2020, GRAVITY Collaboration detected the Schwarzschild precession in the orbit of the S2 star around SMBH at the Galactic Centre

Corrections to the perihelion advances of planets ($''$/ cy) and their real uncertainties.

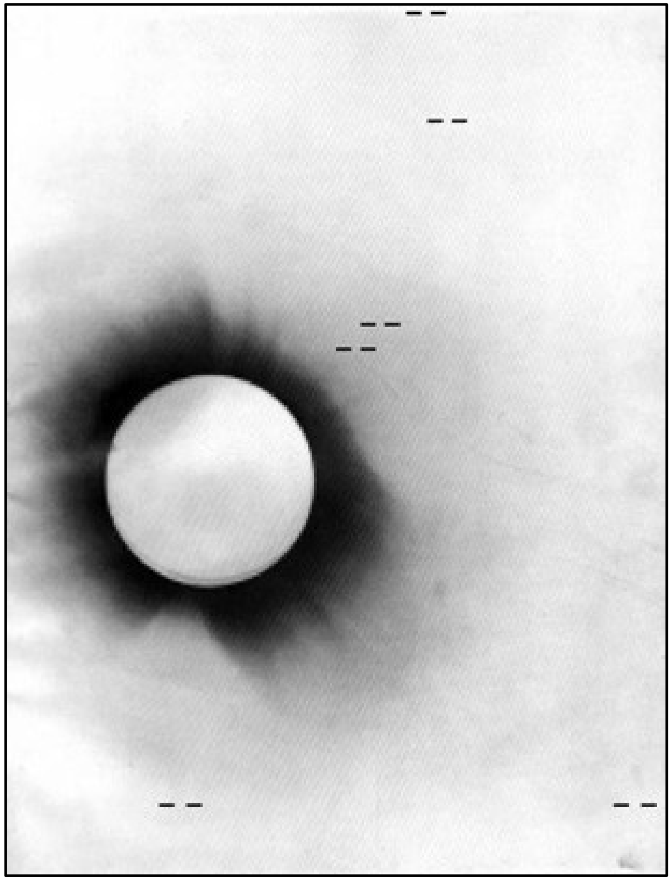
Mercury	Venus	Earth	Mars	Author
42.98	8.62	3.84	1.35	Brumberg, 1972
0.11±0.22 −0.017±0.052 −0.0040±0.0050	−3.03±0.71 — 0.024±0.033	−0.12±0.16 — 0.006±0.007	−0.35±0.24 — −0.007±0.007	Pitjeva, 1986 Pitjeva, 1993 Pitjeva, 2009
Jupiter	**Saturn**	**Uranus**	**Neptune**	**Pluto**
0.067±0.093	−0.010±0.015	− 3.89±3.90	-4.44±5.40	2.84±4.51



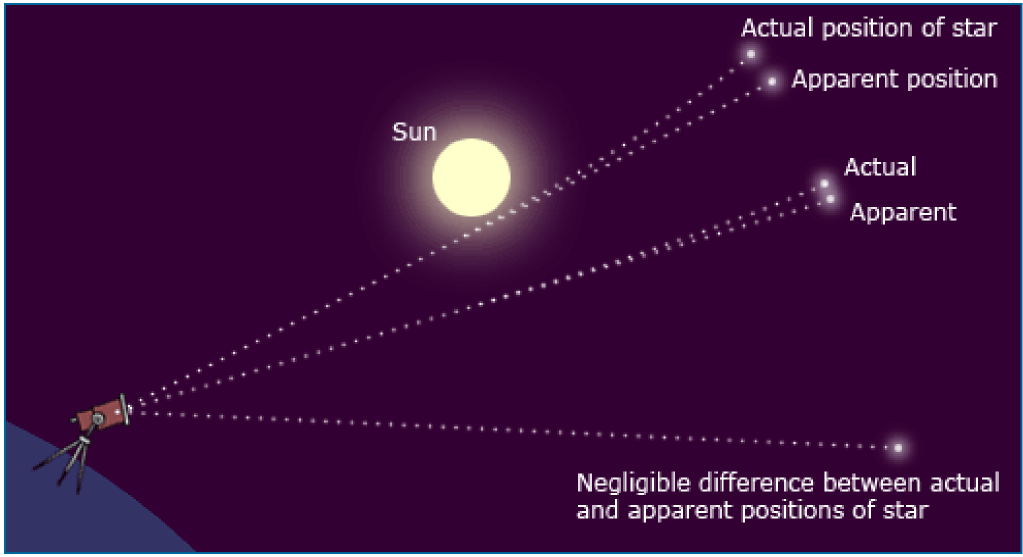
Unbound orbits: deflection of light by the Sun



- Johann Georg von Soldner (1804) - trajectory of particle with speed c in Newtonian gravity deflected by angle:
$$\alpha = \frac{2GM}{c^2\xi}$$
- Albert Einstein (1915) in GR - deflection angle of photons moving along geodesics: $\alpha = \dfrac{4GM}{c^2\xi}$



- Eddington - total solar eclipse in 1919:
  - No light bending: $\alpha = 0''$
  - Newton's mechanics: $\alpha = 0''.87$
  - GR: $\alpha = 1''.75$
- Confirmation of Einstein's predictions:
$\alpha_1 = 1''.98 \pm 0''.12$ $\alpha_2 = 1''.61 \pm 0''.30$
- **Gravitational lens** is a massive celestial object (or a distribution of matter), located between an observer and a distant background source, which gravitational force deflects the light rays from the source



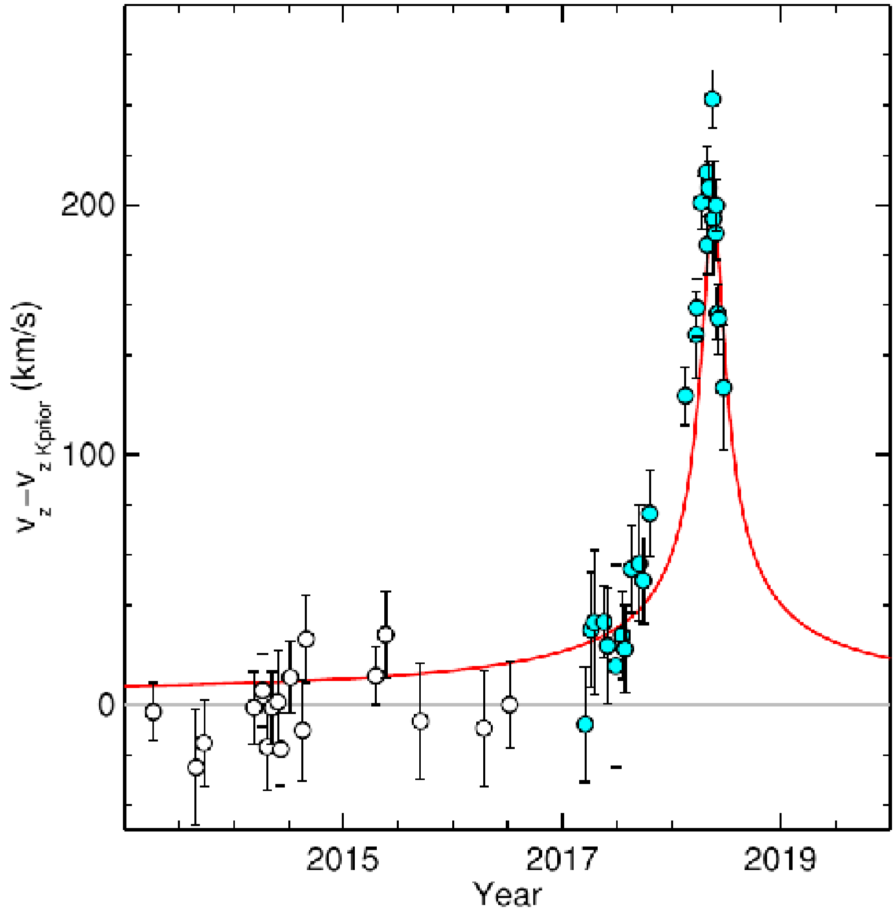
Gravitational redshift of light

- Gravitational redshift z is the shift toward longer wavelengths of electromagnetic radiation emitted by a source in a gravitational field, e.g. at the surface of a massive star
- In spherically symmetric gravitational field, z is:

$$1 + z = \frac{\lambda_\infty}{\lambda_e} = \frac{1}{\sqrt{1 - \frac{R_S}{R_e}}}$$

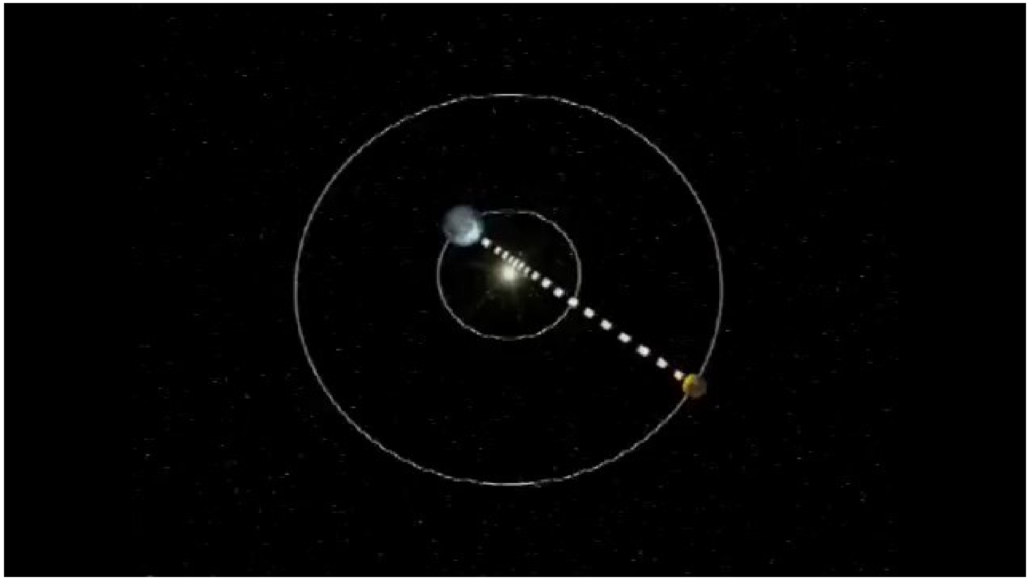
where λ_∞ is the wavelength of the light as measured by the observer at infinity, λ_e is the wavelength measured at the source of emission, and R_e is the radius at which the photon is emitted

- It was identified by astronomical observations in the spectral lines of the star Sirius B, white dwarf 40 Eridani B, Sun, sunlight reflected by the Moon and galaxy clusters
- Pound-Rebka experiment measured frequency shifts in gamma rays as they rose and fell in the gravitational field of the Earth
- Global Positioning System (GPS) must account for the gravitational redshift



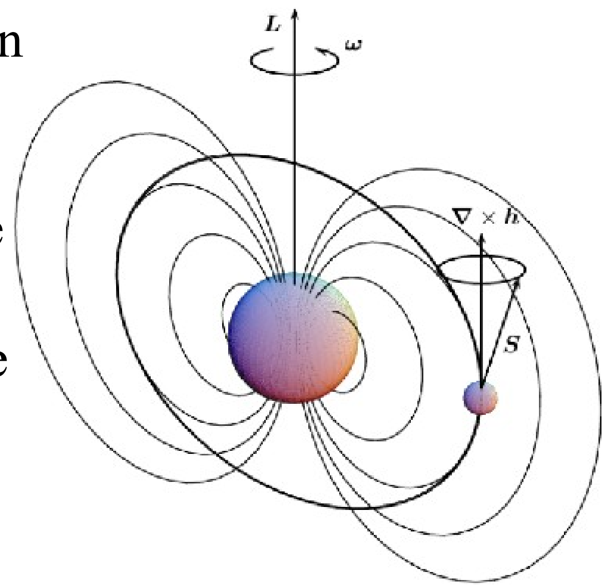
- In 2018, GRAVITY Collaboration detected the gravitational redshift in the orbit of the S2 star around SMBH at the Galactic Centre
- Detected combined transverse Doppler and gravitational redshift up to 200 km/s/c was in agreement with GR predictions

Gravitational (Shapiro) time delay

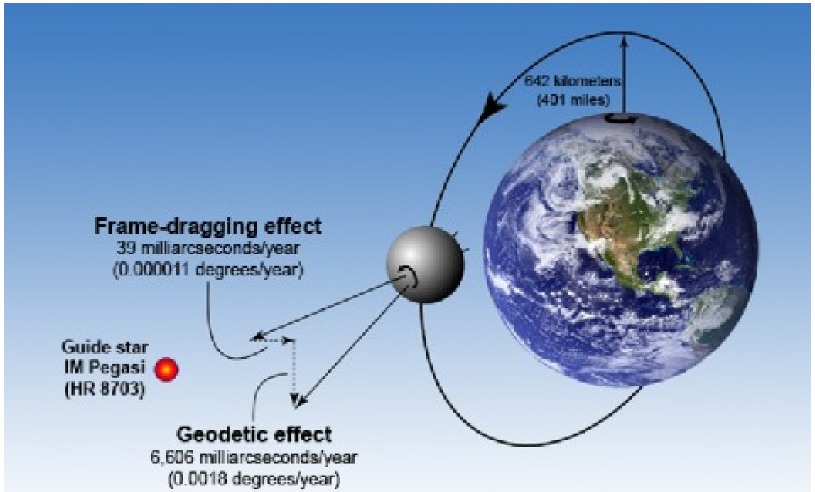


- Radar signals passing near a massive object take slightly longer to travel to a target and longer to return than they would if the massive object was not present
- In the case of a nearly static gravitational field of moderate strength, such as in the case of stars and planets, **Shapiro time delay** is a special case of **gravitational time dilation** (difference of elapsed time between two events as measured by observers situated at varying distances from a gravitating mass)
- Time delay Δt due to light traveling around a massive object is: $c\Delta t = -R_S \ln\left(1 - \vec{R}\cdot\vec{x}\right)$, where $\vec{R}$ is the unit vector pointing from the observer to the source, $\vec{x}$ is the unit vector pointing from the observer to the gravitating mass M, and the dot denotes the usual Euclidean scalar product
- During 1960s, Irwin I. Shapiro proposed and carried out measurements of the time required for radar signals to travel to Venus and Mercury and be reflected back to Earth
- Measured time delay Δt, due to the presence of the Sun, of a radar signal traveling from the Earth to Venus and back, was $\Delta t \sim 200$ μs, matching the time delay predicted by GR
- Shapiro time delay was also confirmed by ranging data of Voyager and Pioneer interplanetary spacecrafts

Precession of orbiting gyroscopes



- Gyroscope is a device used for measuring or maintaining orientation and angular velocity
- In Newtonian gravity, the spin axis of a perfect gyroscope orbiting the Earth would remain forever fixed with respect to absolute space
- In GR, the presence of a large rotating mass, such as Earth or Kerr black hole, causes spacetime to warp (curve) and twist, and thus the spin axis of a perfect gyroscope orbiting the central mass will precess with respect to the distant universe
- Spacetime warping and twisting both cause a precession (at ninety degree angles with respect to one another) of the gyroscopes orbiting the central mass
- The effect caused by warping of spacetime due to the presence of the central mass is called **de Sitter precession (geodetic effect)**
- The effect caused by twisting of spacetime (frame dragging) due to the rotation of the central mass is called **Lense-Thirring precession**
- The total precession is calculated by combining the de Sitter precession with the Lense-Thirring precession
- In 2011 **Gravity Probe B** mission gave direct proof of frame-dragging in the case of the Earth's rotation with a 19% margin of error



	Measured	Predicted
Geodetic precession (mas)	6602 ± 18	6606
Frame-dragging (mas)	37.2 ± 7.2	39.2

Lense-Thirring precession around rotating SMBHs

- A star orbiting a spinning (Kerr) SMBH experiences Lense-Thirring (LT) precession, causing its orbital line of nodes to precess at a rate:

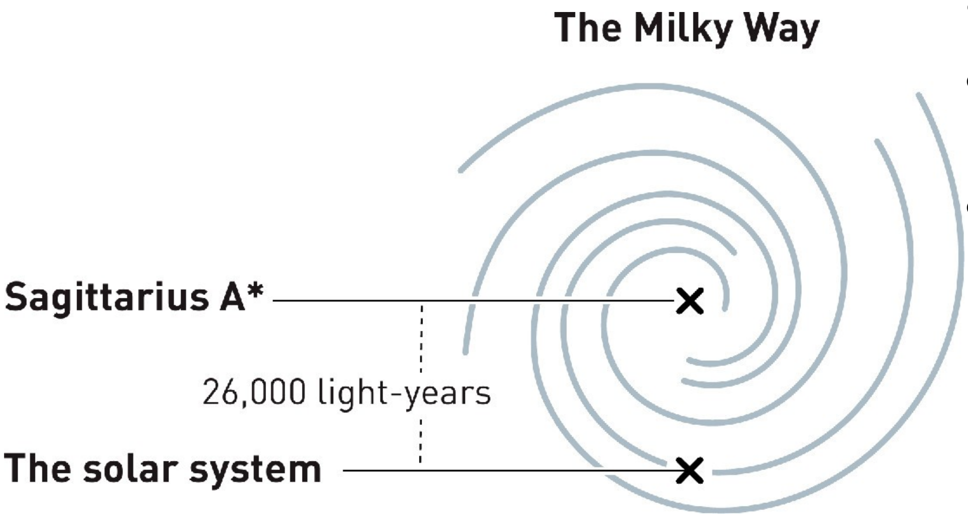
$$\frac{d\Omega}{dt} = \frac{2G^2M^2\chi}{c^3a^3\left(1-e^2\right)^{\frac{3}{2}}},$$

  where a and e are the semimajor axis and eccentricity of the orbit, M is the mass of the SMBH, and χ is the dimensionless spin parameter ($0 < \chi < 1$)
- It should be possible to detect the LT precession of S-stars by long-term monitoring of their orbits around Sgr A*
- The detected LT precession could be then used for constraining the spin χ of Sgr A*

Star	a (mpc)	P (yr)	e	$\dot{\Omega}^{\rm LT}_{\rm max}$ ($''\,{\rm yr}^{-1}$)	$\dot{\Omega}^{\rm LT}_{\rm min}$ ($''\,{\rm yr}^{-1}$)
S4714	4.079 ± 0.012	12.0 ± 0.3	0.985 ± 0.011	8.9	−8.9
S62	3.588 ± 0.02	9.9 ± 0.3	0.976 ± 0.01	5.15	−5.32
S4711	3.002 ± 0.06	7.6 ± 0.3	0.768 ± 0.030	0.35	−0.34

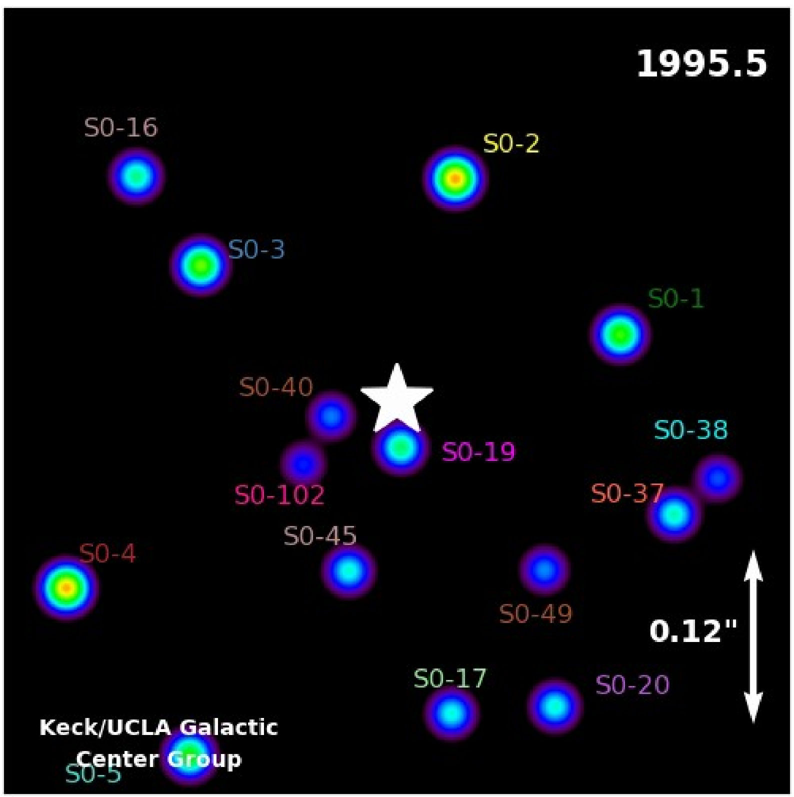
Maximum and minimum values of the LT rate of change of the node Ω of 3 selected short-period S-stars (Iorio, 2020, ApJ, 904, 186)

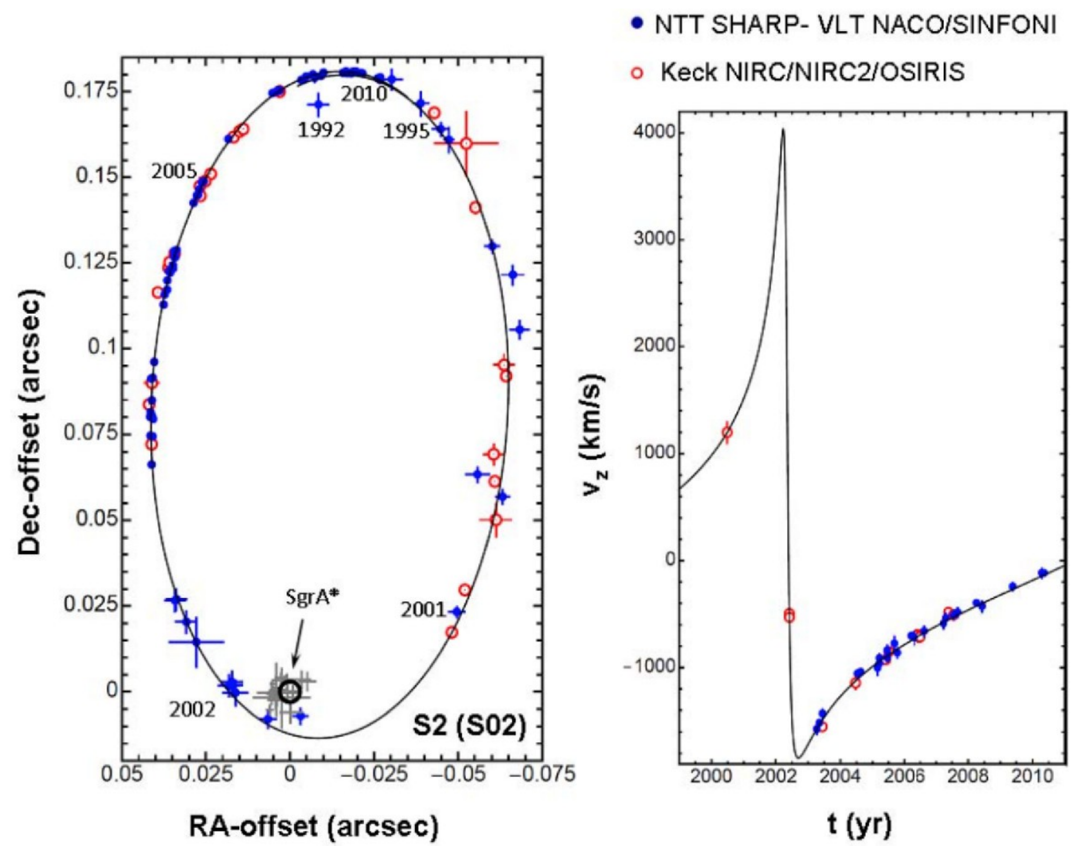
Sgr A*: central SMBH of the Milky Way



Stellar orbits around Sgr A* monitored by 2 groups:

- R. Genzel (ESO): **New Technology Telescope** & **Very Large Telescope**, Chile
- A. Ghez: **Keck telescope**, Hawaii, USA

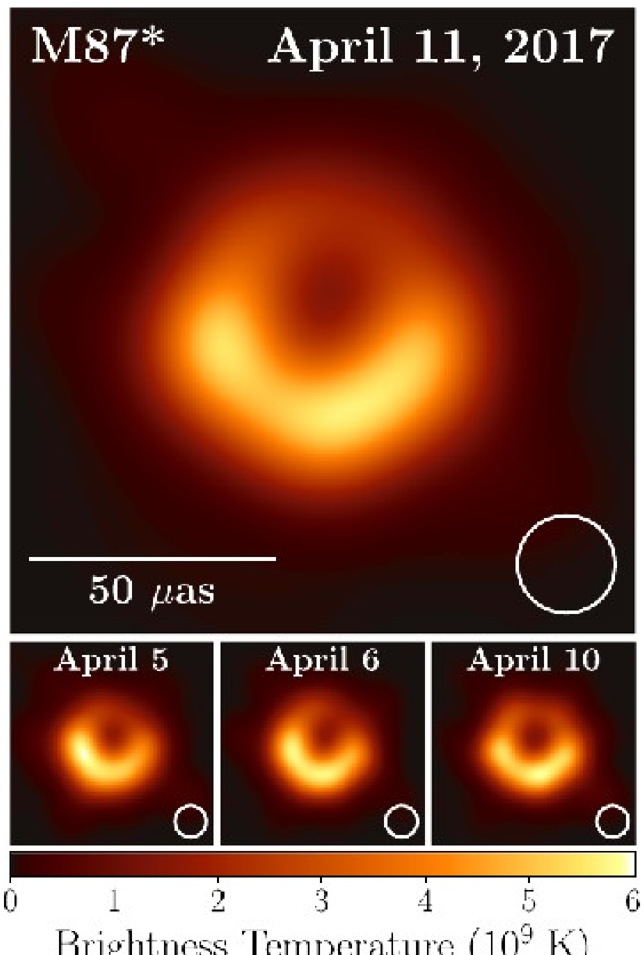


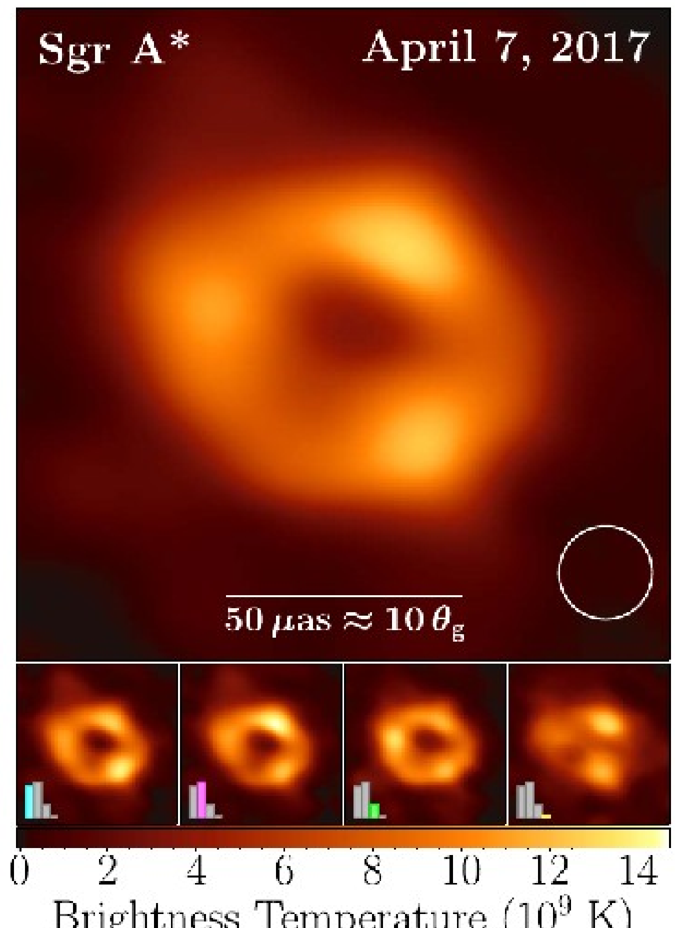


- Nobel Prize in Physics 2020 for the discovery of SMBH with mass of $\approx 4 \times 10^6\, M_\odot$ at Galactic Centre, based on observations of stars orbiting Sgr A*, particularly star S2

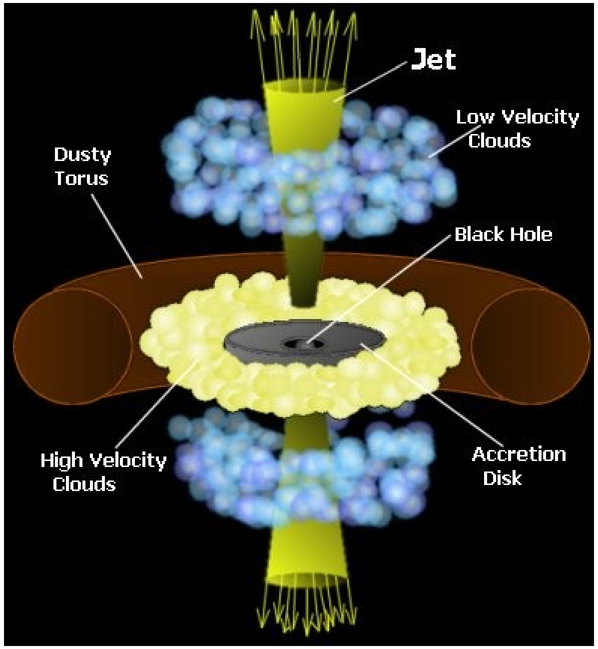
Observed images of the SMBH shadows at the centers of M87 and Milky Way

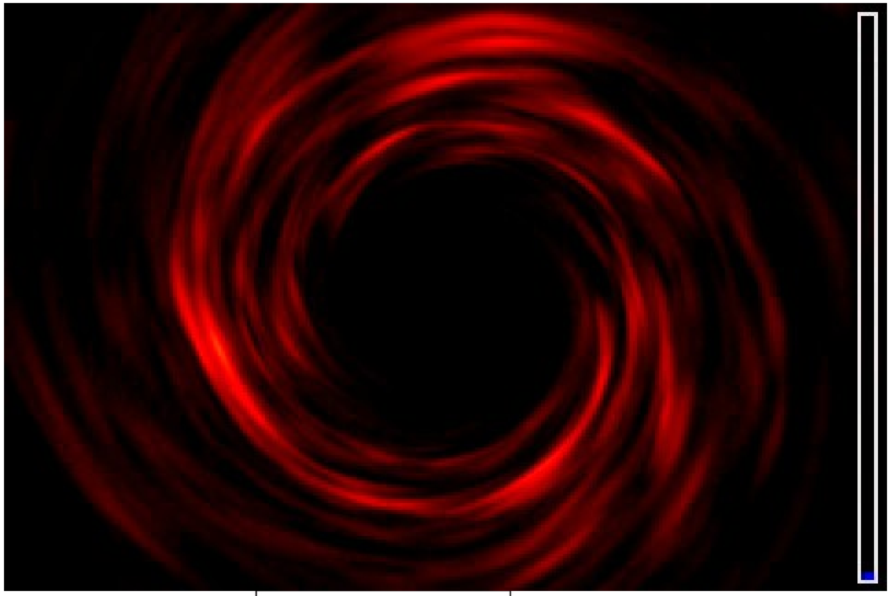
- When surrounded by a transparent emission region, black holes are expected to reveal a dark shadow caused by gravitational light bending and photon capture at the event horizon
- Such shadows were imaged in 2017 by the Event Horizon Telescope (EHT) in the case of central SMBHs of M87 galaxy and the Milky Way
- These images are consistent with expectations for the shadow of a rotating (Kerr) SMBH

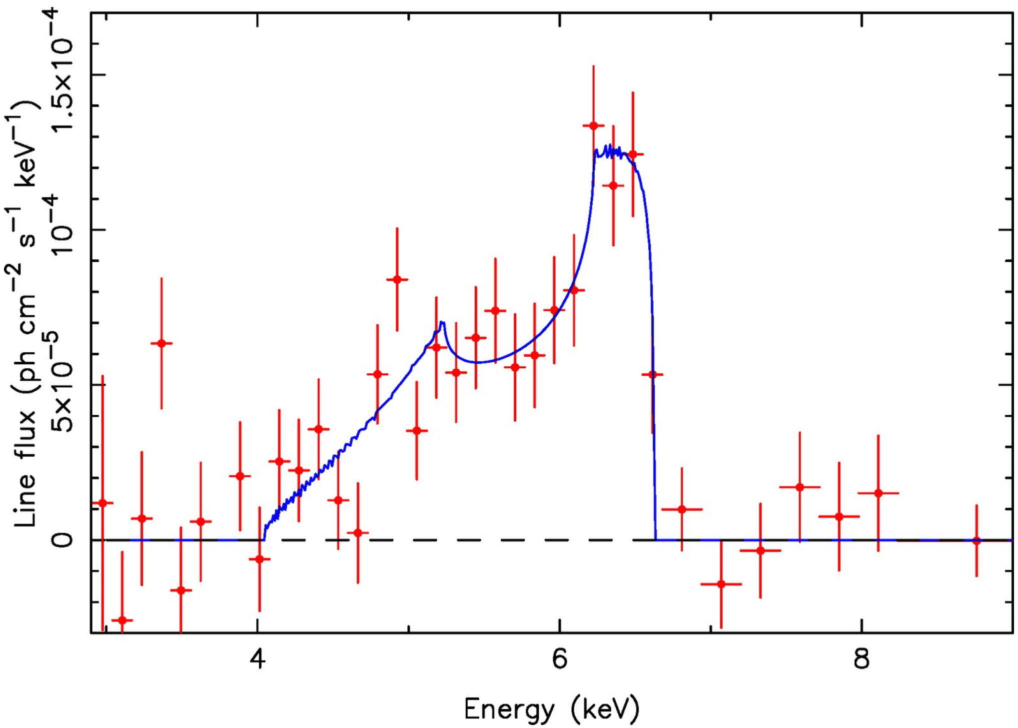


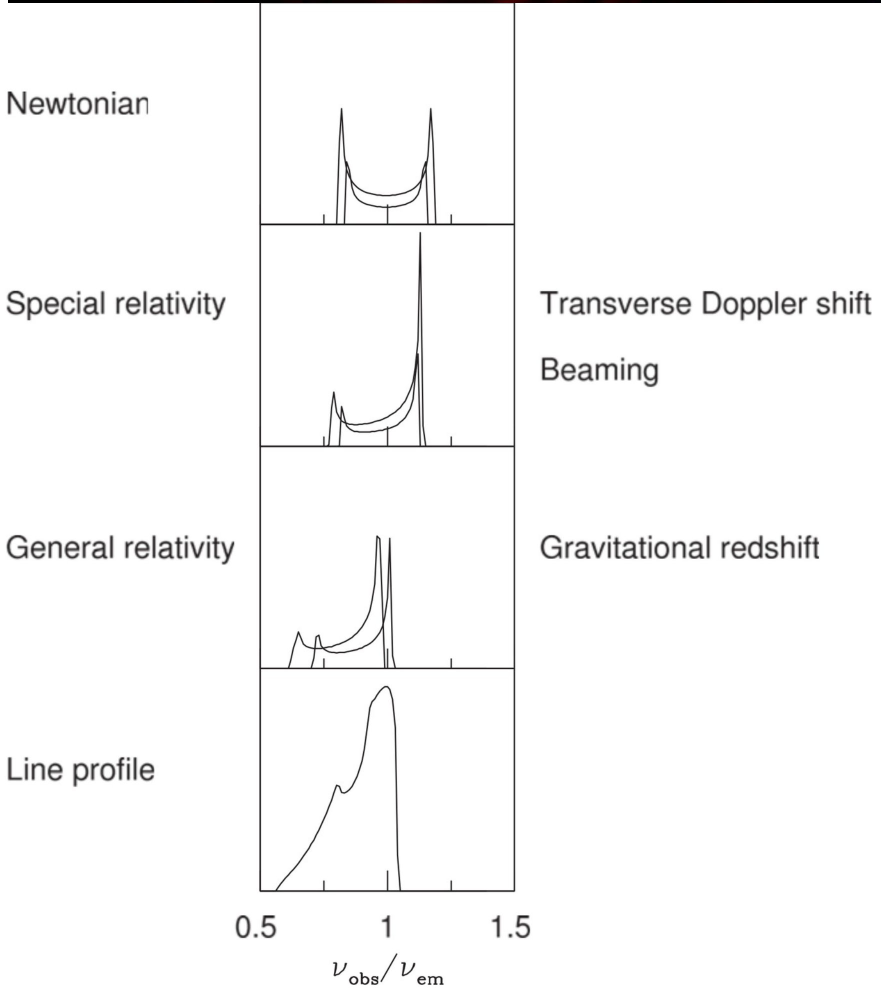


Relativistically broadened spectral lines from AGN



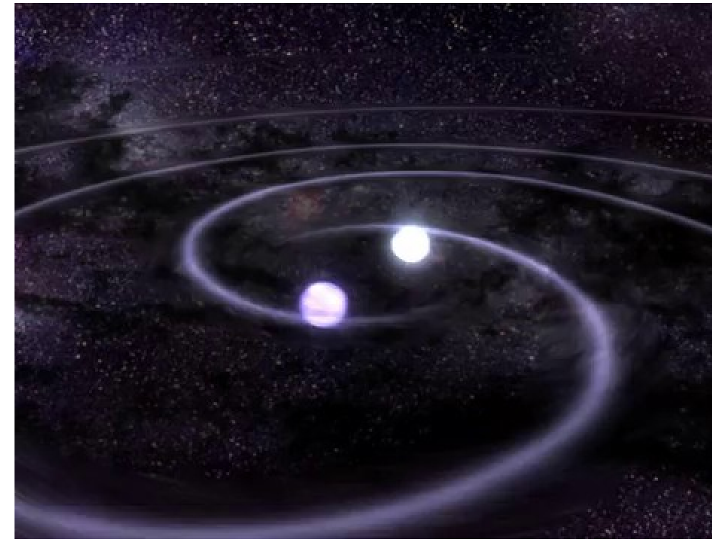




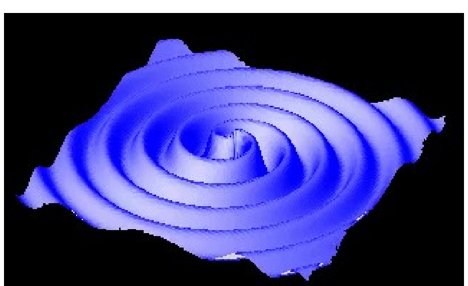


The observed Fe Kα line profile from Seyfert I galaxy MCG-6-30-15, and the modeled profile from a relativistic accretion disk around a Schwarzschild SMBH

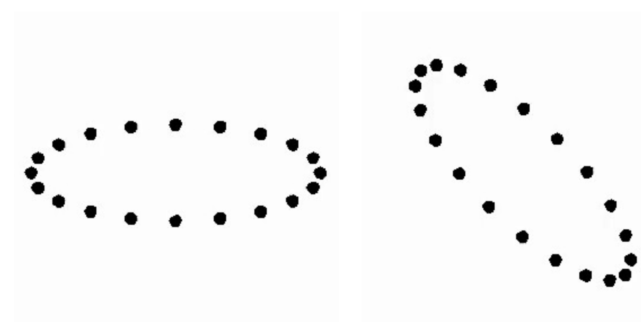
Gravitational waves and their polarizations



- **Gravitational waves** (GW) are ripples in spacetime caused by massive objects moving with extreme accelerations (such as neutron stars or black holes orbiting each other) and disrupting spacetime in such a way that waves of undulating spacetime would propagate in all directions away from the source
- In the frame of GR, Einstein predicted the existence of GW in 1916
- For GW emitted from a binary system at distance R with components of masses m_1 and m_2:
- **GW amplitude** is: $h_0 = \frac{2\eta}{c^4 R}\frac{(Gm)^2}{p}$, where $\eta = \frac{m_1 m_2}{(m_1+m_2)^2}$ is symmetric mass ratio of the system, $p = a(1 - e^2)$ is the orbit's semilatus rectum and $m = m_1 + m_2$
- **GW polarizations** are: $h_+ = h_0\, H_+, \quad h_\times = h_0\, H_\times$
- H_+ and $H_\times$ are scale-free polarizations, which for a circular orbit are given by: $H_+ = -(1+\cos^2 i)\cos 2(\Omega\tau + \omega),\ H_\times = -2\cos i \sin 2(\Omega\tau+\omega),$ where $\Omega = \sqrt{Gm/p^3}$ and $\tau = t - R/c$ is retarded time

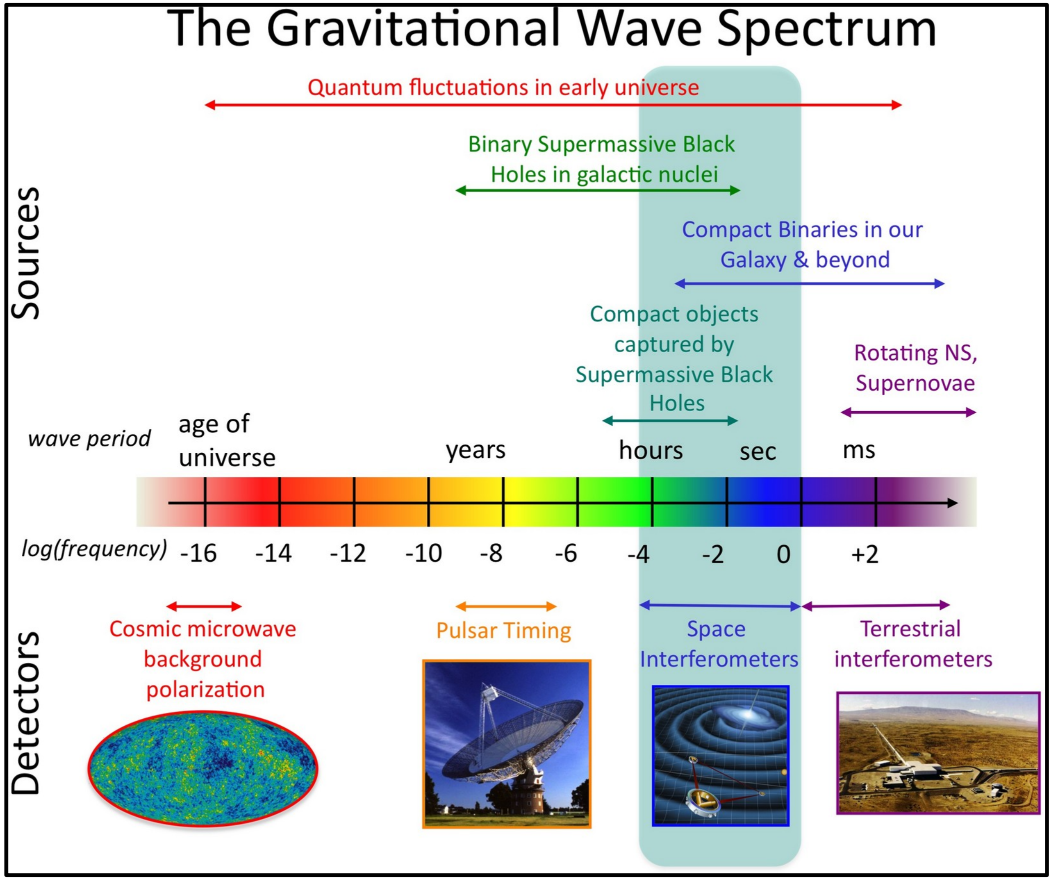


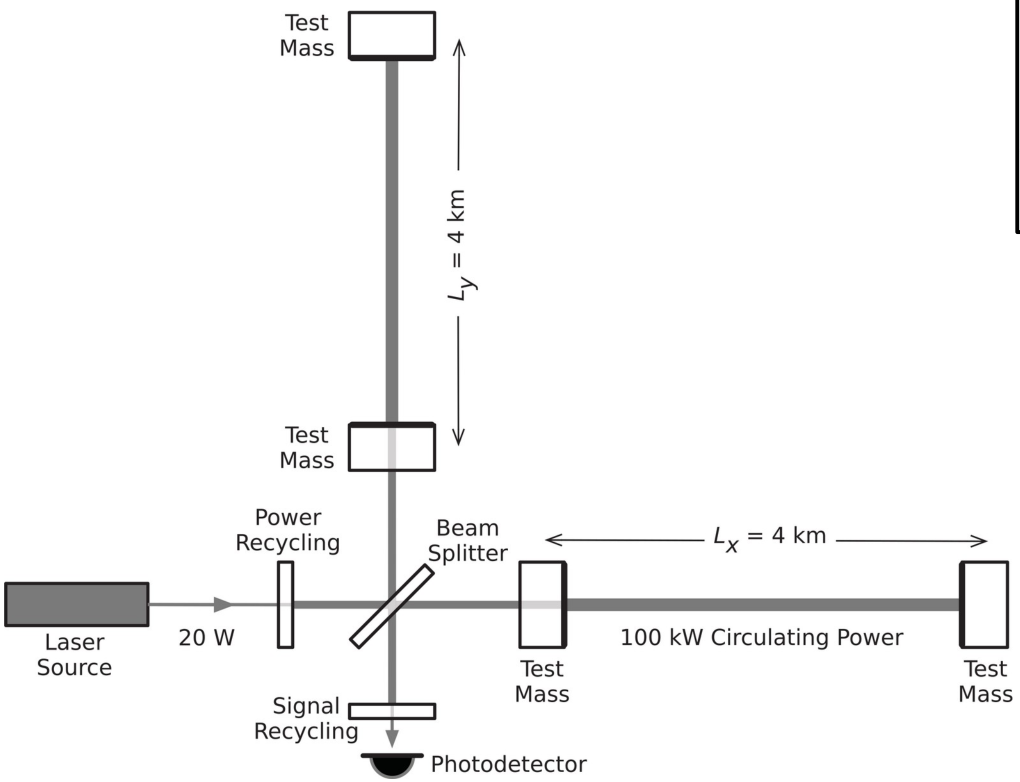
The effects of a plus-polarized (left) and cross-polarized (right) GW on a ring of particles



Astrophysical sources of GWs

- Possible astrophysical sources of GW:
  - binary systems of white dwarfs, neutron stars and black holes
  - supernovae
  - cosmic inflation
- Depending on their frequency, GWs could be detected by ground-based (LIGO) or space-based (LISA) interferometers, as well as by pulsar timing arrays

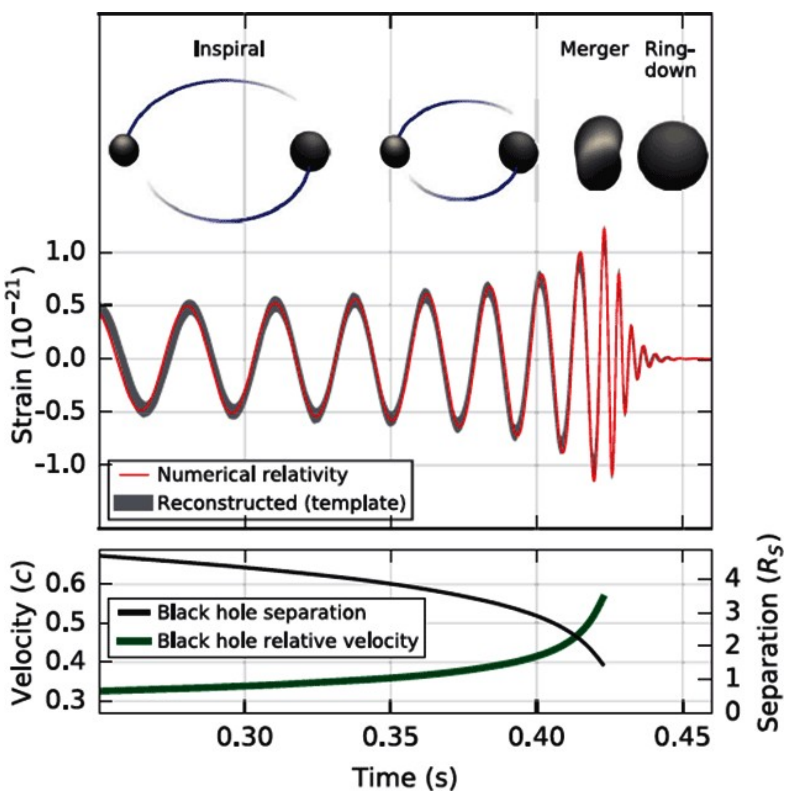


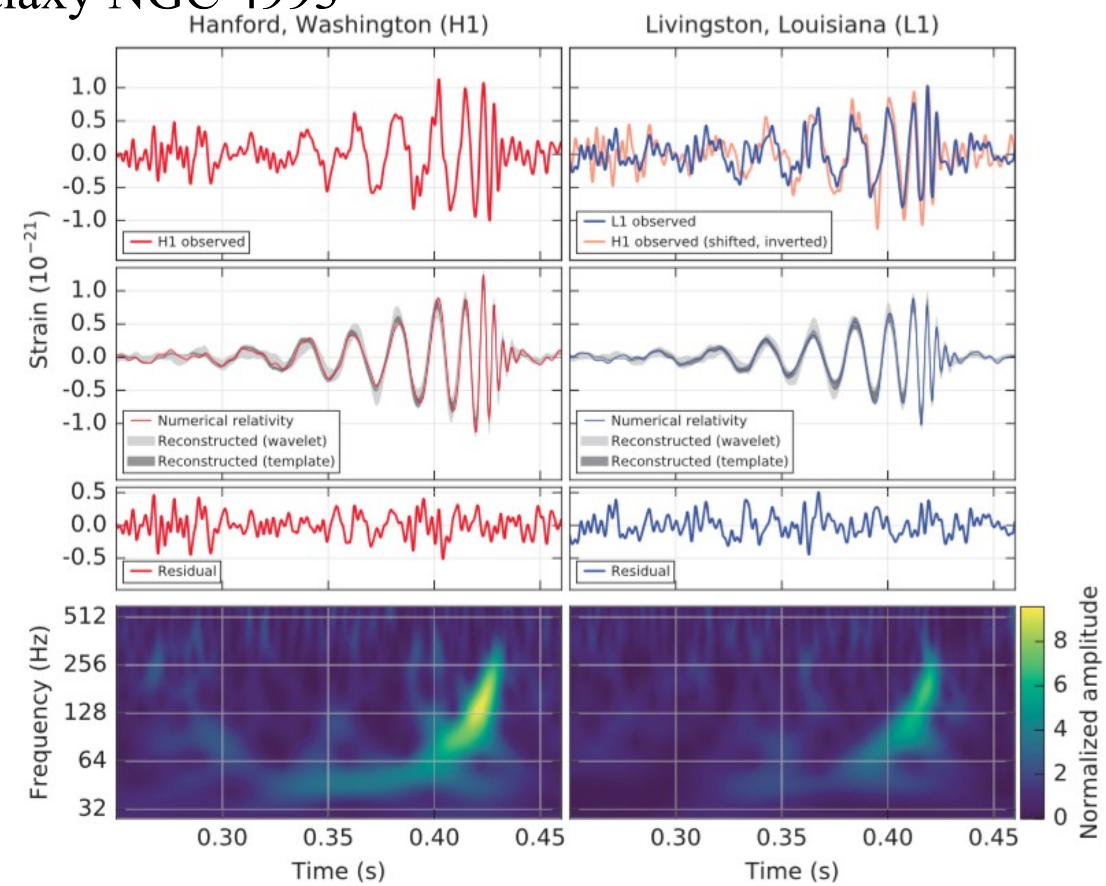


- LIGO consists of two L-shaped detectors with 4 km long vacuum chambers
- It is capable of detecting a change in distance between its mirrors 1/10,000th the width of a proton, which is equivalent to measuring the distance to the nearest star (some 4.2 light years away) to an accuracy smaller than the width of a human hair

Detection of GWs

- On September 14, 2015, LIGO/Virgo Collaborations detected the first GW generated by two colliding black holes 1.3 billion light-years away
- Detected signal was in excellent agreement with the corresponding simulated waveform obtained by numerical relativity
- Nobel Prize in Physics 2017 was awarded to Rainer Weiss, Kip Thorne and Barry Barish for their role in the direct detection of GWs
- In 2017 LIGO/Virgo and Fermi/INTEGRAL made the joint detection of GW and γ-rays from a binary neutron star merger in the galaxy NGC 4993





Exam questions

1. Vacuum solutions to the field equations: Schwarzschild, Kerr, Reissner-Nordström and Kerr-Newman metric, black holes, perihelion precession of Mercury's orbit, deflection of light by the Sun, gravitational redshift of light, gravitational (Shapiro) time delay
2. Precession of orbiting gyroscopes (Lense-Thirring effect), discovery of the central SMBH of the Milky Way, observed images of the SMBH shadows at the centers of M87 and our Galaxy, relativistically broadened spectral lines from Active Galactic Nuclei, gravitational waves and their detection

Literature

- Textbooks:

1. Weinberg, S., 1972, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley-VCH
2. Poisson, E. and Will, C. M., 2014, *Gravity: Newtonian, Post-Newtonian, Relativistic*, Cambridge University Press

- Papers:

1. Event Horizon Telescope Collaboration, Akiyama, K., et al. 2022, ApJL, 930, L12
2. Event Horizon Telescope Collaboration, Akiyama, K., et al. 2019, ApJL, 875, L1
3. Everitt, C. W. F., et al. 2011, PhRvL, 106, 221101
4. GRAVITY Collaboration, Abuter, R., et al. 2020, A&A, 636, L5
5. GRAVITY Collaboration, Abuter, R., et al. 2018, A&A, 615, L15
6. Jovanović, P. 2012, NewAR, 56, 37
7. LIGO Scientific Collaboration and Virgo Collaboration, Fermi and INTEGRAL, Abbott, B. P., et al. 2017, ApJL, 848, L13
8. LIGO Scientific Collaboration and Virgo Collaboration, Abbott, B. P., et al. 2016, PhRvL, 116, 061102

Exercise 1

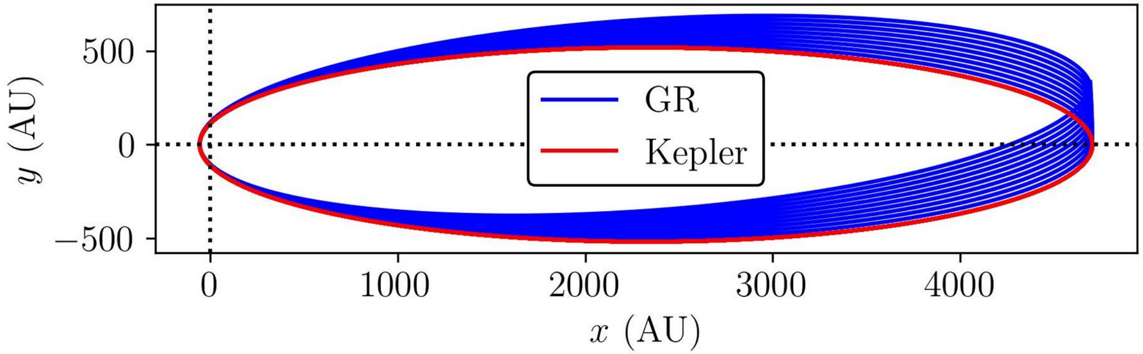
- Use the provided Python program "orbit.py" for two-body problem in General Relativity to simulate the orbits of the specified S-stars around Sgr A* (the central SMBH of the Milky Way) with mass of 4.3 million solar masses.

  a) simulate the orbits of S14, S62 and S4714 star during 10 orbital periods, knowing that their semi-major axes a and eccentricities e are:
     - S14: a = 2382 AU, e = 0.9761
     - S62: a = 740 AU, e = 0.976 and
     - S4714: a = 841 AU, e = 0.985

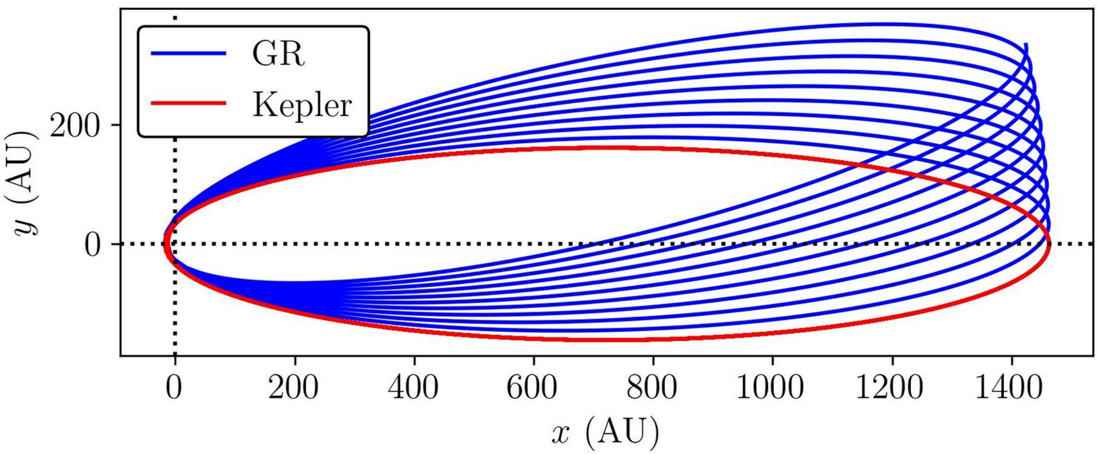
  b) simulate the orbit of S4714 star during 150 orbital periods and discuss the shapes of relativistic orbits

Solution 1

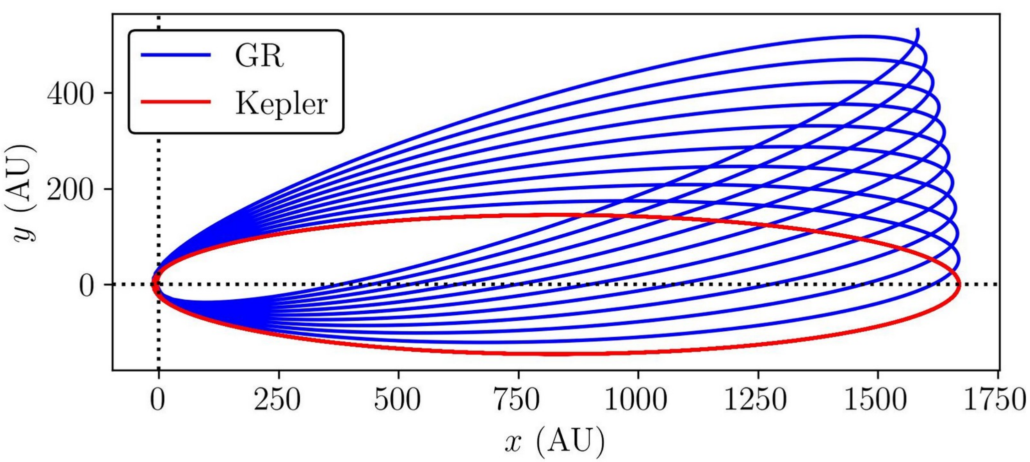
S14

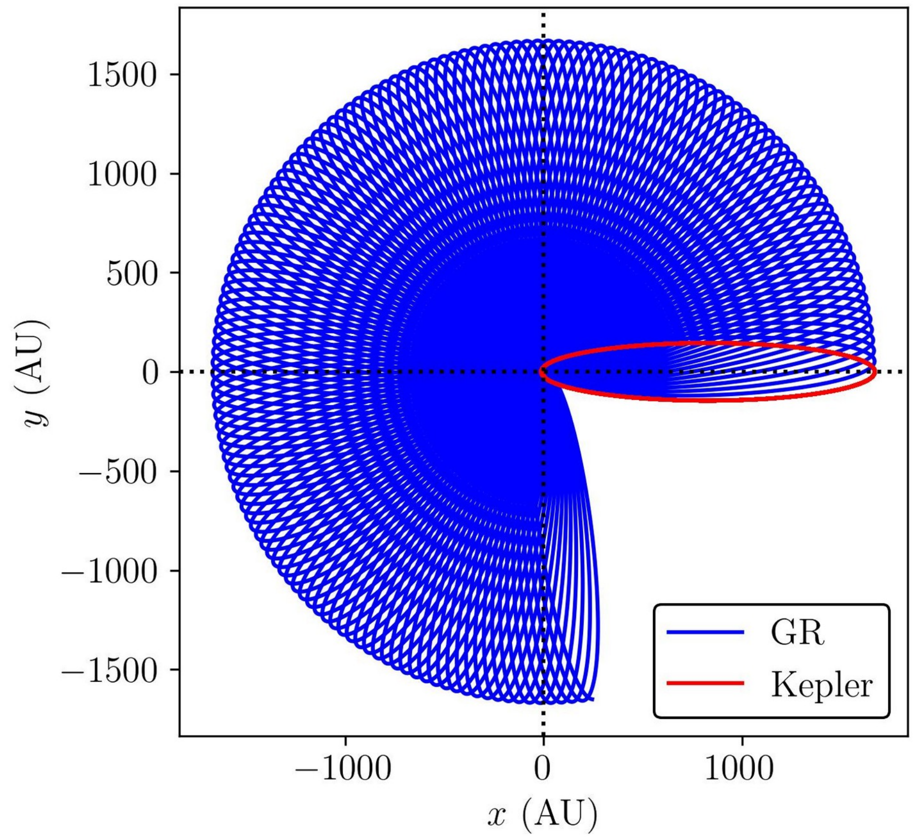


S62



S4714





b) Two-body orbits in GR are rosette-shaped