MASS 2023 Course:
Gravitation and Cosmology

Predrag Jovanović
Astronomical Observatory Belgrade

Lecture 09

- Brief introduction to Big Bang cosmology
- Cosmological principle
- Comoving frame
- Friedmann-Lemaître-Robertson-Walker metric
- Scale factor of the universe
  - Cosmological redshift
  - Hubble and deceleration parameters
- Perfect fluid and cosmological equation of state
- Relativistic (Friedmann) cosmology
  - Friedmann equations
  - Cosmological density parameters: Ω_M, Ω_Λ and Ω_κ
- Standard ΛCDM cosmological model
- Exercises

Brief introduction to cosmology I

- **Cosmology:** scientific discipline that studies the origin, evolution, dynamics, large-scale structure, and ultimate fate of the Universe
- **Static Universe** was the first scientific cosmological model based on Newton's law of gravity in which the Universe is both spatially and temporally infinite, space is flat (or Euclidean) and neither expanding nor contracting
- Until the early 1920s, most astronomers thought that the Milky Way contained all the stars in the Universe, and other galaxies were considered to be nebulae within our Galaxy
- In 1912 Vesto Slipher discovered that most spiral nebulae had considerable redshifts, and thus "positive" (recessional) velocities
- In 1924 Edwin Hubble's measurement of the great distances to the nearest spiral nebulae showed that they were indeed other galaxies
- In 1927 Georges Lemaître proposed that the inferred recession of the nebulae was due to the expansion of the Universe

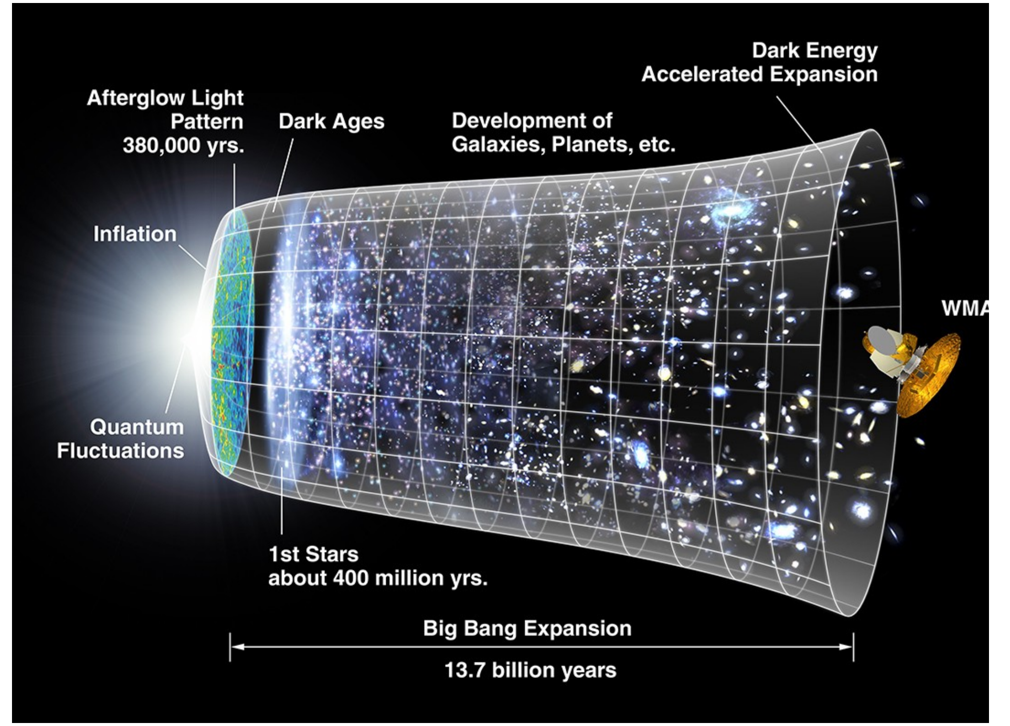
Brief introduction to cosmology II

- In 1929 Edwin Hubble discovered an approximate relationship between the redshifts of such galaxies and the distances to them
- These discoveries are today considered strong evidence for an expanding Universe and the Big Bang theory
- In 1922 Alexander Friedmann derived a set of equations that govern the expansion of space in homogeneous and isotropic models of the Universe within the context of Einstein's General Relativity
- **Big Bang cosmology:** currently accepted theory according to which, some 13.8 billion years ago Universe was in an extremely hot and dense state which expanded rapidly. After this initial expansion from a singularity, the Universe cooled and presently is in continuously expanding state
- **ΛCDM model:** the standard model of Big Bang cosmology according to which, the Universe is mostly composed of ***dark energy*** (represented by cosmological constant Λ) and ***cold dark matter*** (CDM)

Important stages in the evolution of the Universe

1. Expansion of early universe is described by quantum field theory:

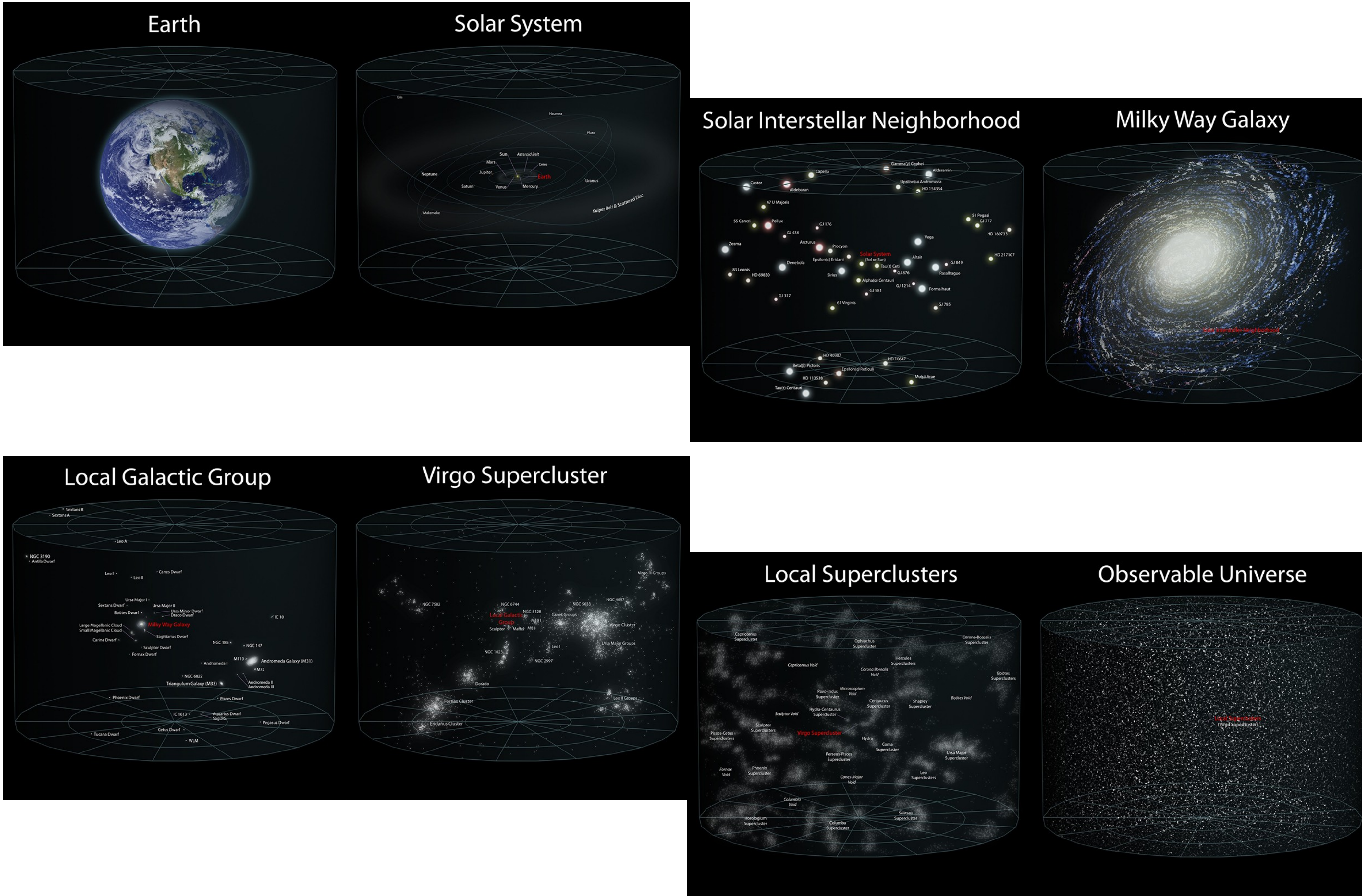
- **Planck epoch** ($< 10^{-43}$ s after the Big Bang): temperature of over 10^{32} °C (Planck Temperature) and all four fundamental forces are unified
- **Grand unification epoch** ($10^{-43} – 10^{-36}$ s): forces separate from each other
- **Inflationary epoch** ($10^{-33} – 10^{-32}$ s): rapid expansion due to the **inflaton** field



2. After the end of inflation, expansion of universe is described by **General relativity**:

- **Recombination/Decoupling** (from 240,000 to 300,000 years): temperature falls to 3,000 K, hydrogen and helium ions capture and bound electrons ("recombination"), so the universe becomes transparent to light ("photon decoupling"), making this the earliest epoch observable today (CMBR)
- **Dark Ages** (from 300,000 to 150 million years): period dominated by dark matter
- **Star and Galaxy Formation** (300 - 500 million years onwards): density irregularities in primordial gas cause it to collapse and heat enough to trigger nuclear fusion between hydrogen atoms, and thus to create the first stars. Large volumes of matter collapse and form galaxies which then form groups, clusters and superclusters

Observable universe

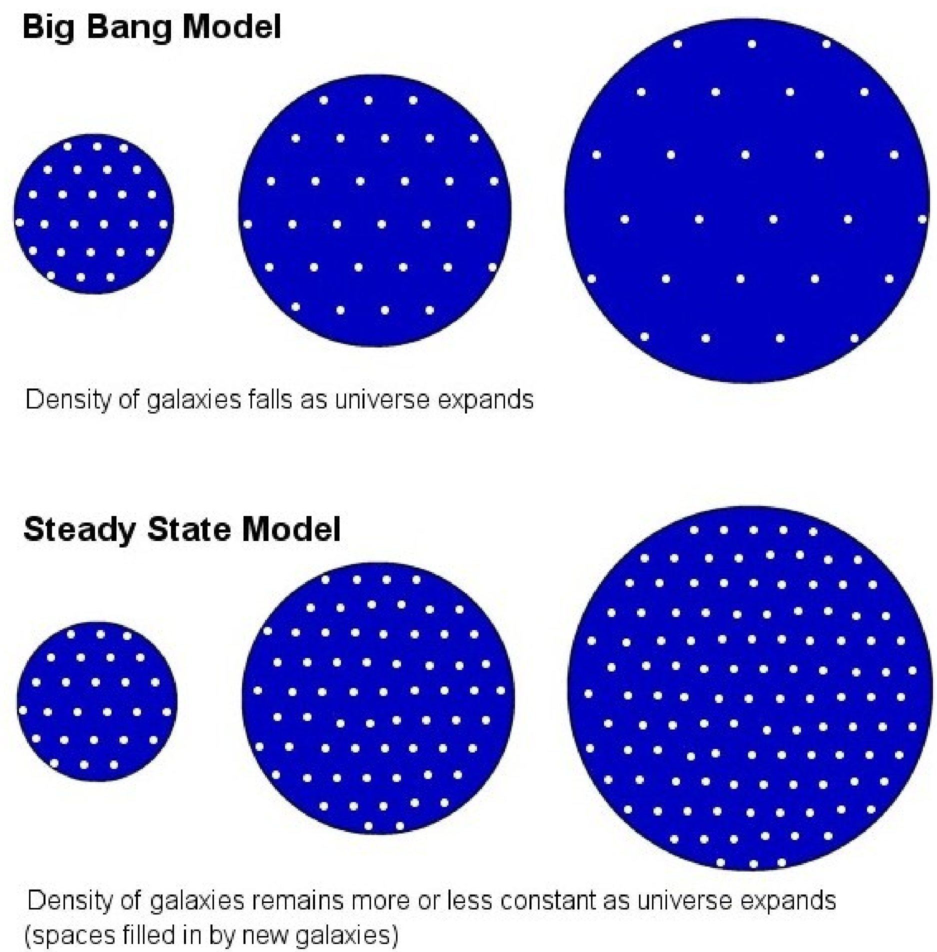


Cosmological principle

At large scales the Universe is:

1. **homogeneous:** there is no preferred observing position (no special locations such as center)
2. **isotropic:** there is no preferred observing direction (no special directions such as an axis)

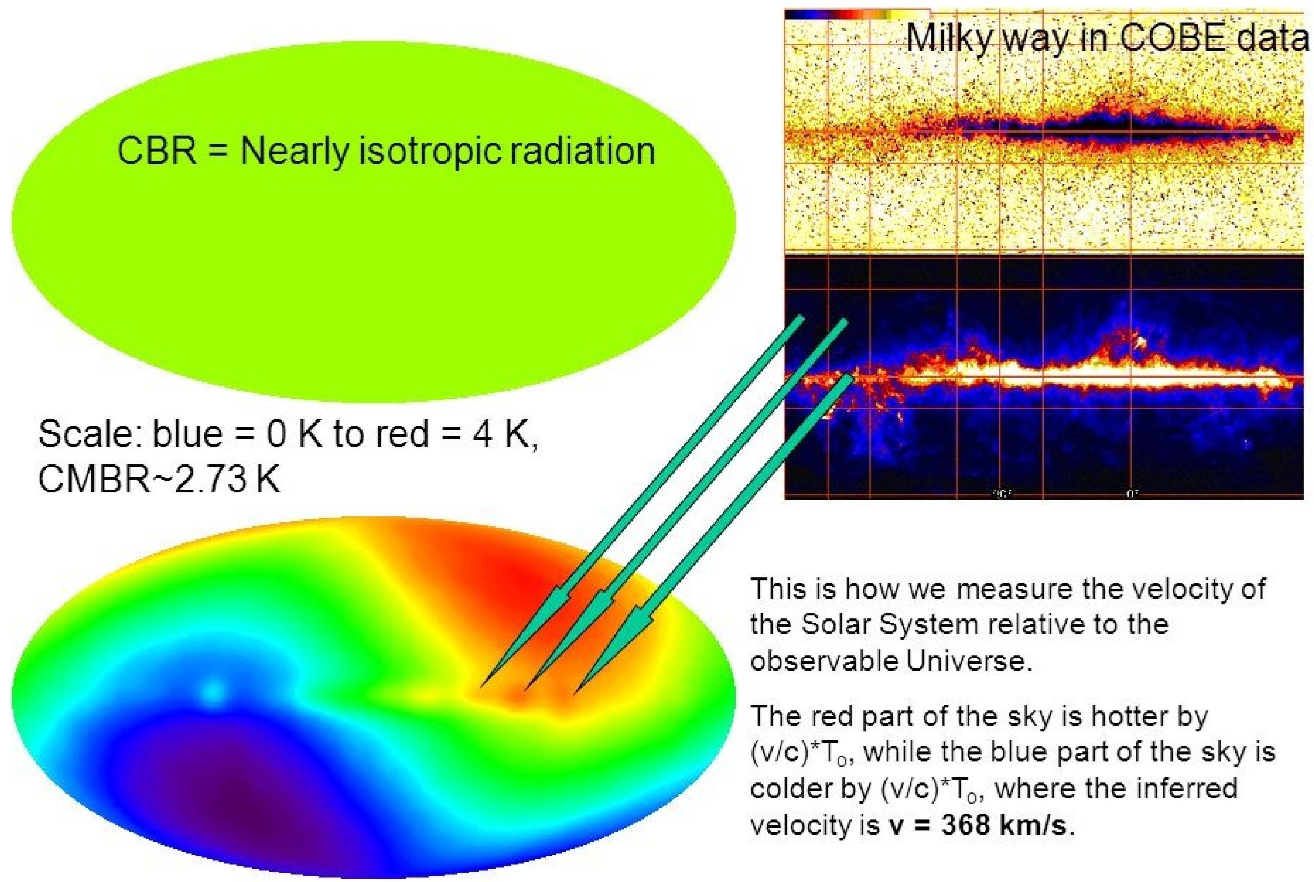
- Big Bang cosmology is based on **cosmological principle**
- It was preceded by **Steady State theory** based on the **perfect cosmological principle** which states that the Universe is homogeneous and isotropic in space and time, i.e. the same everywhere and always



Comoving frame

- A special local reference frame defined by isotropy of CMBR
- Natural coordinate system which assigns constant spatial coordinates to observers who perceive the universe as isotropic

- **Comoving observers** move along with the Hubble flow
- **Non-comoving observers** do not perceive the Universe as isotropic (including CMBR), but they see regions which are systematically blueshifted or redshifted
- **Peculiar velocity of the observer:** the velocity of an observer relative to the local comoving frame



Blue 2.724 K to red for 2.732 K.
This is the dipole component of CBR.

Spacetime geometry at cosmological scales

- Metric in **static universe**, the first relativistic cosmological model introduced by Einstein, has the following form due to the cosmological principle: $ds^2 = c^2 dt^2 - dl^2$, where *dl* is the spatial distance on a 3-dimensional surface of constant Riemannian curvature (such as e.g. a three-sphere), embedded in a flat 4-dimensional space
- *dl* is time independent, and in a flat 4-dimensional space with Cartesian coordinates *x*, *y*, *z, w*:

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2$$

- Three-sphere (3-dimensional analog of the 2-dimensional surface of a balloon) is the set of points (*x*, *y*, *z*, *w*) at fixed distance *R* from the origin: $x^2 + y^2 + z^2 + w^2 = R^2$
- The fourth coordinate, *w*, is thus given by: $w^2 = R^2 - r^2, \quad r^2 = x^2 + y^2 + z^2 \;\Rightarrow$

$$dw = -\frac{r\,dr}{w} = -\frac{r\,dr}{\sqrt{R^2 - r^2}} \;\Rightarrow\; dl^2 = dx^2 + dy^2 + dz^2 + \frac{r^2 dr^2}{R^2 - r^2}$$

- In the spherical coordinates $x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$:

$$dl^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \frac{r^2 dr^2}{R^2 - r^2} = \frac{dr^2}{1 - r^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

- Generalization to an arbitrary spatial curvature *κ*: $dl^2 = \dfrac{dr^2}{1 - \kappa\, r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$
- In the case of an **expanding (or contracting) universe**, the spatial part of the metric *dl* scales with time by a universal function of time *a*(*t*), and then the full metric is given by:

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1 - \kappa\, r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Friedmann–Lemaître–Robertson–Walker (FLRW) metric

- The most general metric which describes a geometry of a homogeneous, isotropic, expanding or contracting Universe:

$$ds^2 = c^2 dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$$

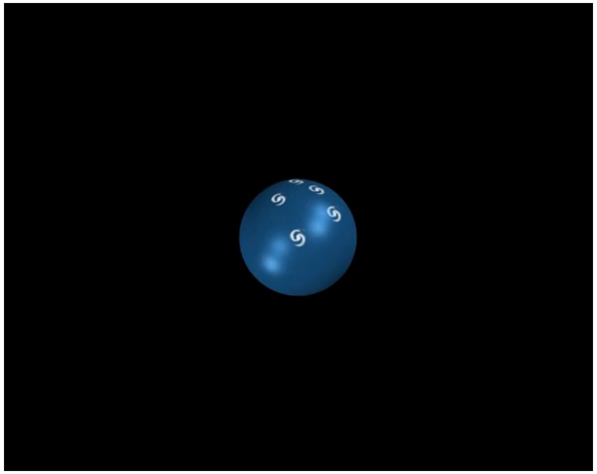
$$a(t) = \frac{R(t)}{R_0} \wedge \kappa = \frac{k}{R_0^2} \quad \Rightarrow \quad \boxed{ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right]}$$

$$\boxed{ds^2 = c^2 dt^2 - R^2(t)\left[d\chi^2 + S_k^2(\chi) d\Omega^2\right], \ r = S_k(\chi), \ S_k(\chi) = \begin{cases} \sin\chi, & k = +1 \\ \chi, & k = 0 \\ \sinh\chi, & k = -1 \end{cases}}$$

- r, φ, θ - **comoving coordinates**
- k and κ - constants representing the **spatial curvature**:

1. $k = -1, 0, +1$ for negative, zero, and positive spatial curvature respectively, and then $R(t)$ is in units of length

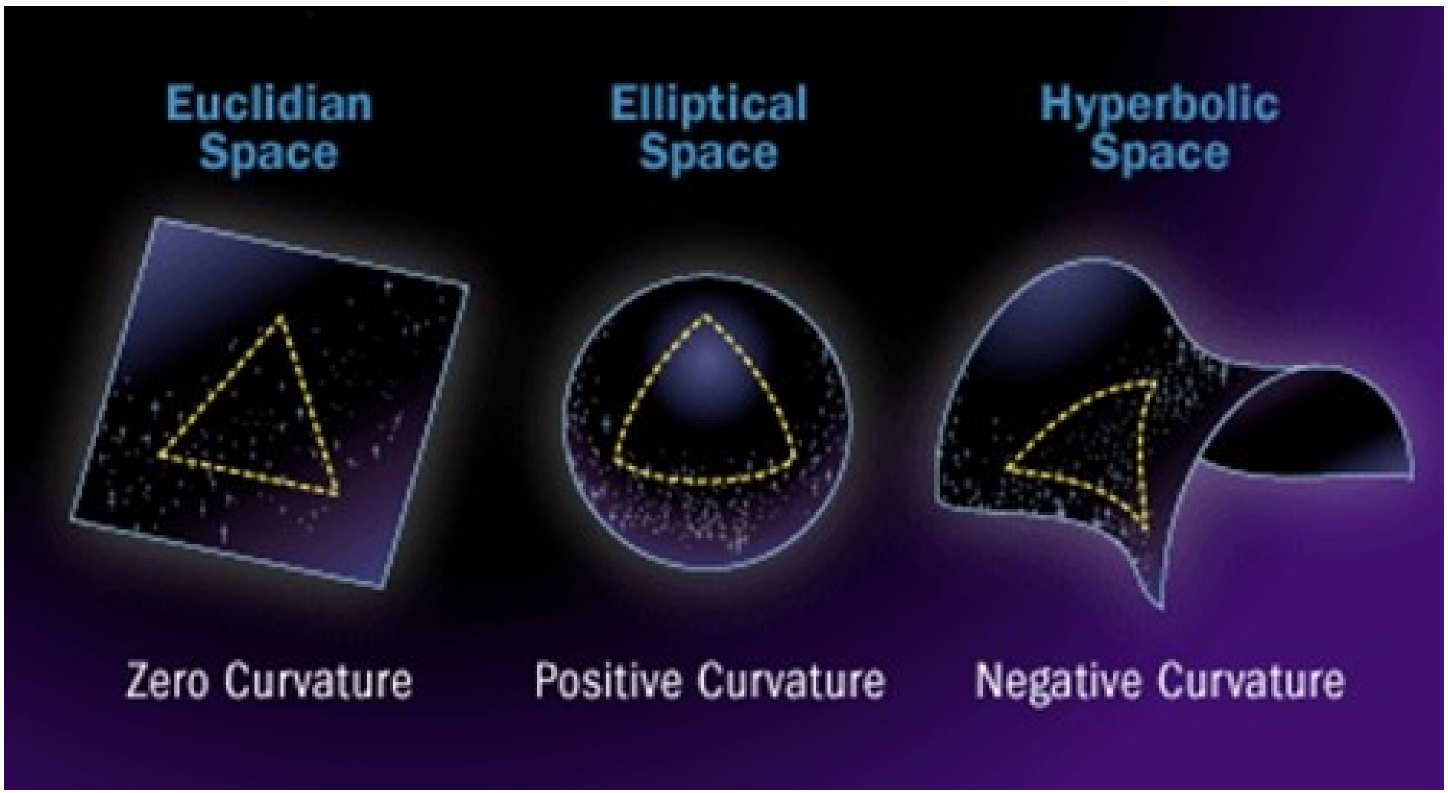
2. κ is the spatial curvature in units of length^{-2} at the time when $a(t) = 1$, and then $a(t)$ is dimensionless



- $R(t)$ and $a(t)$ - **scale factor** ("the size") of the Universe

Spatial geometry

- $\kappa > 0$: closed space-time with spherical spatial geometry
- $\kappa = 0$: flat space-time with Euclidean spatial geometry
- $\kappa < 0$: open space-time with hyperbolic spatial geometry



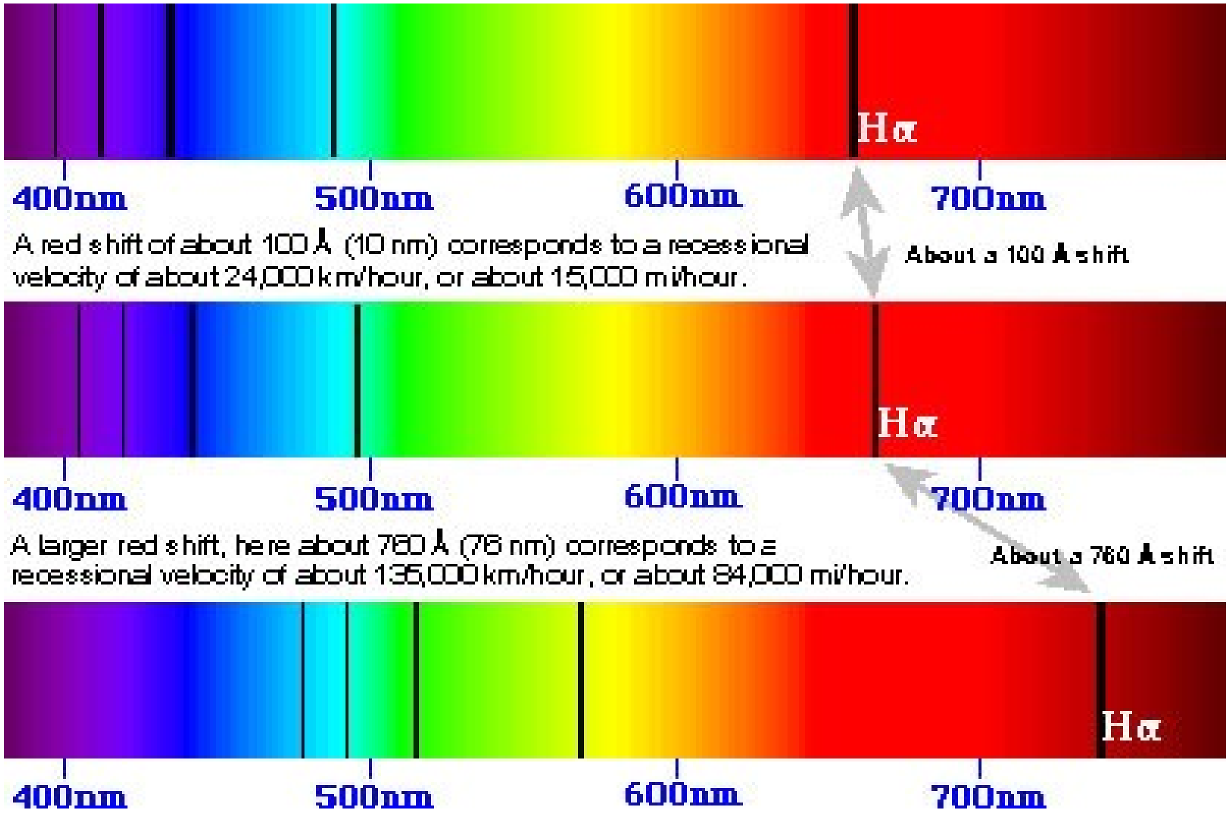
- **<u>Friedmann (Lemaître) cosmological models</u>**: $a(t)$ and κ are obtained from the field equations of GR

Cosmological redshift

- Some fundamental cosmological quantities are defined by scale factor

1. **Cosmological redshift** (Lemaître's definition):

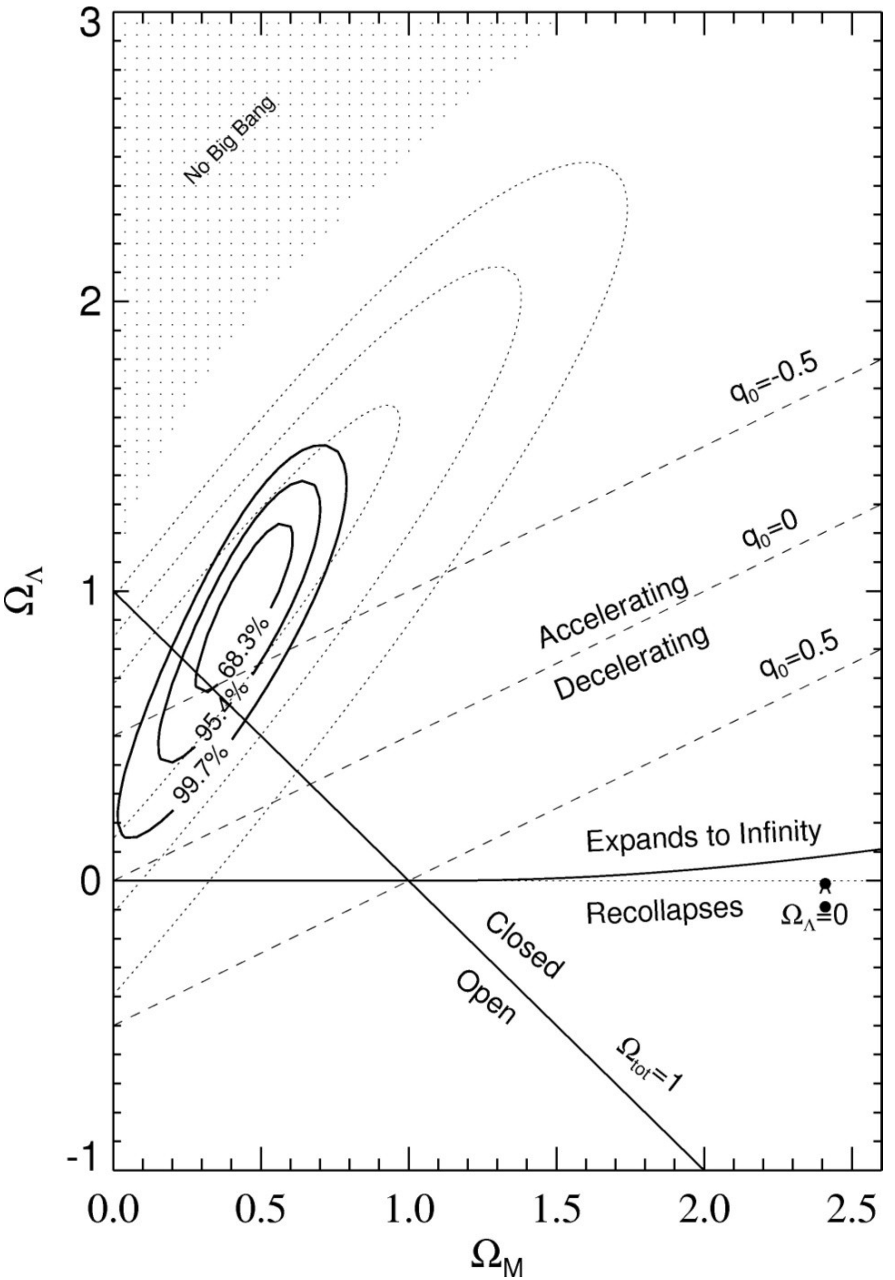
$$1 + z = \frac{R(t_0)}{R(t_e)} = \frac{1}{a(t_e)} = \frac{\lambda_0}{\lambda_e} = 1 + \frac{\overbrace{\lambda_0 - \lambda_e}^{\Delta\lambda}}{\lambda_e}$$



Hubble parameter (constant)

$$H(t) := \frac{\dot{a}}{a}, \quad H_0 = H(t_0)$$

- H > 0 – expanding Universe
- *H < 0* – contracting Universe
- **Hubble time:** $t_H := 1/H_0$
- **Hubble distance:** $D_H := c/H_0$



Riess et al. 2004, ApJ, 607, 665

Deceleration parameter

$$q := -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2}$$

- *q* > 0 – deceleration
- q < 0 – acceleration

- Nobel Prize in Physics 2011: **accelerating expansion of the Universe**

Ricci tensor and scalar for FLRW metric

1. Nonzero components of FLRW metric tensor:

$$g_{00} = 1;\; g_{11} = -\frac{a^2}{1-\kappa r^2};\; g_{22} = -a^2 r^2;\; g_{33} = -a^2 r^2 \sin^2\theta \quad \Rightarrow \quad g^{\mu\mu} = \frac{1}{g_{\mu\mu}}$$

2. $\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\eta}\left(\frac{\partial g_{\eta\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\eta\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\eta}}\right) \Rightarrow$ nonzero Christoffel symbols:

$$\Gamma^{0}_{11} = \frac{a\dot{a}}{1-\kappa r^2};\quad \Gamma^{0}_{22} = a\dot{a}r^2;\quad \Gamma^{0}_{33} = a\dot{a}r^2\sin^2\theta;$$

$$\Gamma^{1}_{10} = \Gamma^{1}_{01} = \Gamma^{2}_{20} = \Gamma^{2}_{02} = \Gamma^{3}_{30} = \Gamma^{3}_{03} = \frac{\dot{a}}{a};\quad \Gamma^{3}_{32} = \Gamma^{3}_{23} = \cot\theta;$$

$$\Gamma^{1}_{11} = \frac{\kappa r}{1-\kappa r^2};\quad \Gamma^{1}_{22} = -r(1-\kappa r^2);\quad \Gamma^{1}_{33} = -r(1-\kappa r^2)\sin^2\theta;$$

$$\Gamma^{2}_{21} = \Gamma^{2}_{12} = \Gamma^{3}_{31} = \Gamma^{3}_{13} = \frac{1}{r};\quad \Gamma^{2}_{33} = -\sin\theta\cos\theta.$$

3. Ricci tensor: $R_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu} \Rightarrow$ nonzero components:

$$R_{00} = -3\frac{\ddot{a}}{a};\quad R_{11} = \frac{\left(\ddot{a}a + 2\dot{a}^2 + 2\kappa\right)}{1-\kappa r^2};$$

$$R_{22} = r^2\left(\ddot{a}a + 2\dot{a}^2 + 2\kappa\right);\quad R_{33} = r^2\left(\ddot{a}a + 2\dot{a}^2 + 2\kappa\right)\sin^2\theta$$

4. Ricci scalar: $R = g^{\mu\nu}R_{\mu\nu} = -\frac{6}{a^2}\left(\ddot{a}a + \dot{a}^2 + \kappa\right)$

Perfect fluid and cosmological equation of state

- At cosmological scales, matter and radiation are assumed to have continuous distribution, so that they can be approximated by **perfect fluid**
- Perfect fluid can be completely characterized by its rest frame **energy density** $\boldsymbol{\rho_e}$ and isotropic **pressure** $\boldsymbol{p}$, which are proportional up to a dimensionless number $\boldsymbol{\omega}$ (**cosmological equation of state**): $\boxed{\omega = \frac{p}{\rho_e}}$, $\rho_e = c^2\rho$
- Equation of state ω has different values for different media (fluids) with positive ρ:
  - pressureless matter (ordinary or dark): $\omega = 0$ (e.g. cold dust)
  - ultra-relativistic matter: $\omega = 1/3$ (e.g. radiation, matter in the early universe)
  - dark energy with negative pressure (cosmological constant): $\omega = -1$
  - curvature : $\omega = -1/3$
- Equation of state $\Rightarrow$ evolution of universe depends only on ρ: $\boxed{\rho \propto a^{-3(1+\omega)}}$
- For non-relativistic matter: $\rho \propto a^{-3}$
- For radiation: $\rho \propto a^{-4}$
- Energy-momentum tensor for perfect fluid:
  $T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu + p\, g^{\mu\nu}$

Equation of State (w)	
Radiation	1/3
Matter (pressureless)	0
Curvature	-1/3
Cosmological Constant	-1
Matter (general)	0<w<1/3
Quintessence	-1<w<-1/3

- The perfect fluid is at rest with respect to comoving frame: $U^\mu = (c, 0, 0, 0) \Rightarrow$ the nonzero components of $T^{\mu\nu}$ are: $T_{00} = c^2\rho, \quad T_{ii} = -p g_{ii}$

Friedmann equations

- The field equations of GR in the case of perfect fluid and FLRW metric reduce to two independent **Friedmann equations**
- By substituting the previous results for $g_{\mu\nu}, R_{\mu\nu}, R, T_{\mu\nu}$ into the field equations:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \Rightarrow$$

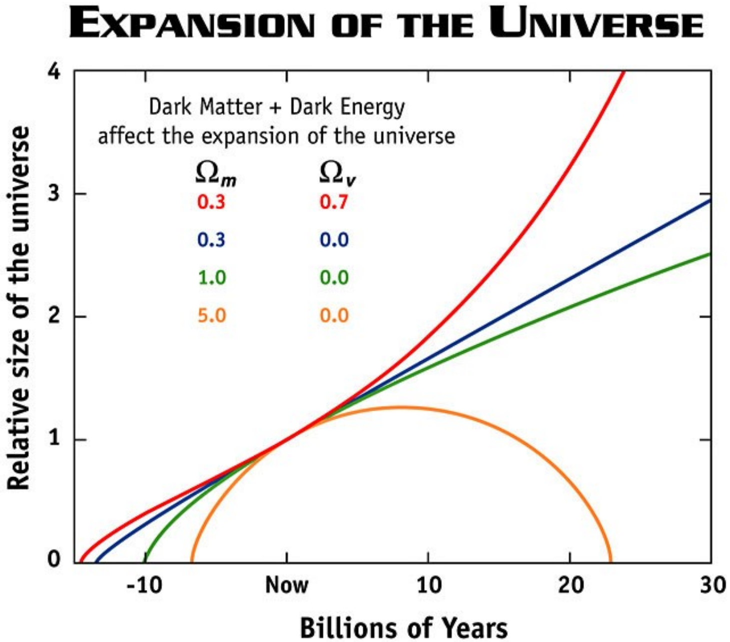
- For $\mu\nu = ii \quad \Rightarrow$ $$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{a^2} + \frac{\Lambda\, c^2}{3}}$$
- For $\mu\nu = 00 \quad \Rightarrow$ $$\boxed{\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda\, c^2}{3}}$$
- Friedmann equations describe dynamics of homogeneous, isotropic expanding or contracting universe
- They were derived by Alexander Friedmann in 1922

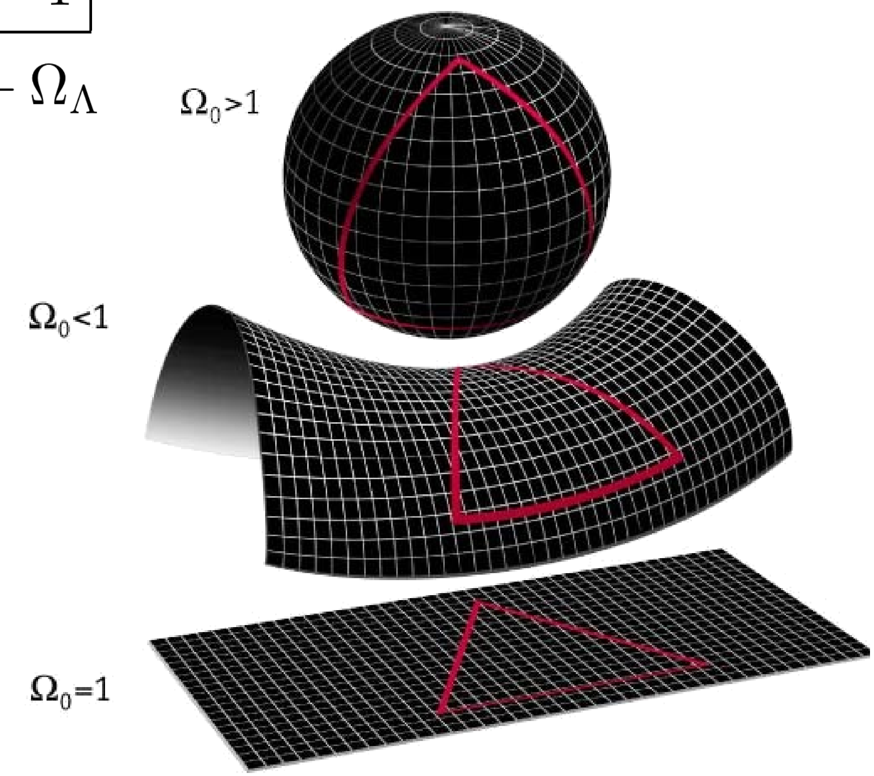
Cosmological dimensionless density parameters

- **Critical density**: 1st Friedmann equation for present epoch $\overset{\kappa=0}{\underset{\Lambda=0}{\Rightarrow}} \rho_c = \dfrac{3H_0^2}{8\pi G}$
- **<u>Dimensionless density parameters</u>**:

$$\boxed{\Omega_M = \frac{\rho_0}{\rho_c} = \frac{8\pi G\rho_0}{3H_0^2}; \quad \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}; \quad \Omega_\kappa = -\frac{\kappa c^2}{H_0^2}} \Rightarrow$$

- 1st Friedmann equation: $\left(\frac{\dot{a}}{a}\right)^2 = H^2 = H_0^2\left(\Omega_M \frac{\rho}{\rho_0} + \frac{\Omega_\kappa}{a^2} + \Omega_\Lambda\right)$
- At present epoch it reduces to: $\boxed{\Omega_M + \Omega_\Lambda + \Omega_\kappa = 1}$
- Mass-energy budget of the universe: $\Omega_0 = \Omega_M + \Omega_\Lambda$





Standard ΛCDM cosmological model

- The simplest Friedmann cosmological model which is in good agreement with the observed cosmic microwave background radiation (CMBR), accelerating expansion, abundances of light elements and the distribution of galaxies at large scales
- **Cosmological parameters:** Hubble constant (H_0) and dimensionless density parameters of: matter (Ω_M), dark energy (Ω_Λ) and spatial curvature (Ω_κ)
- ΛCDM became standard cosmological model after the discovery of CMBR in 1964
- Regions of the universe for $z \lesssim 1000$ are dominated by matter for which $p/c^2 \ll \rho$, so **1st Friedmann equation** becomes:

$$\rho = \frac{\rho_0}{a^3} \;\Rightarrow\; \left(\frac{\dot{a}}{a}\right)^2 = H^2 = H_0^2\left(\frac{\Omega_M}{a^3} + \frac{\Omega_\kappa}{a^2} + \Omega_\Lambda\right) \;\wedge\; a = \frac{1}{1+z} \;\Rightarrow$$

- **Hubble parameter as function of *z*:**

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{z}}{1+z}\right)^2 = H_0^2\left[\Omega_M(1+z)^3 + \Omega_\kappa(1+z)^2 + \Omega_\Lambda\right]}$$

- Similarly, the **2nd Friedmann equation**: $\boxed{\frac{\ddot{a}}{a} = H_0^2\left[\Omega_\Lambda - \frac{1}{2}\Omega_M(1+z)^3\right]}$

Values of cosmological parameters

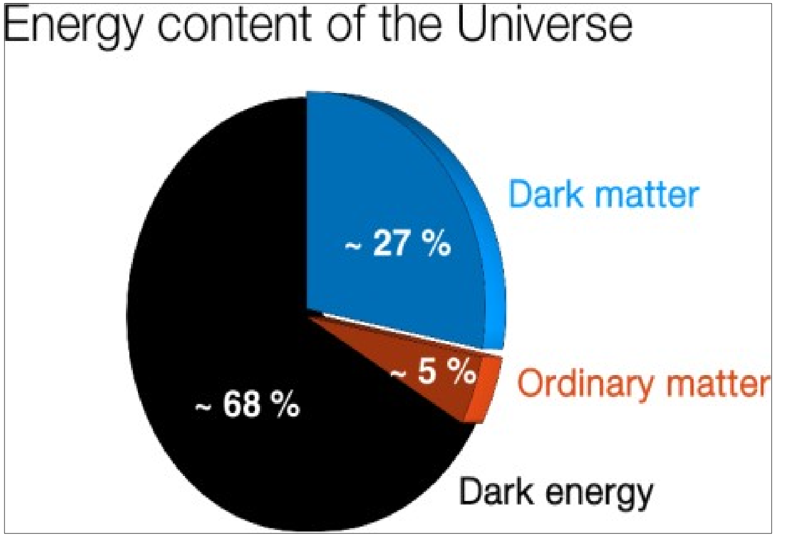
WMAP Seven-year Cosmological Parameter Summary

Description	Symbol	*WMAP*-only	*WMAP*+BAO+H_0
Parameters for the Standard ΛCDM Model[a]			
Age of universe	t_0	13.75 ± 0.13 Gyr	13.75 ± 0.11 Gyr
Hubble constant	H_0	71.0 ± 2.5 km s^{-1} Mpc^{-1}	$70.4^{+1.3}_{-1.4}$ km s^{-1} Mpc^{-1}
Baryon density	Ω_b	0.0449 ± 0.0028	0.0456 ± 0.0016
Physical baryon density	$\Omega_b h^2$	$0.02258^{+0.00057}_{-0.00056}$	0.02260 ± 0.00053
Dark matter density	Ω_c	0.222 ± 0.026	0.227 ± 0.014
Physical dark matter density	$\Omega_c h^2$	0.1109 ± 0.0056	0.1123 ± 0.0035
Dark energy density	Ω_Λ	0.734 ± 0.029	$0.728^{+0.015}_{-0.016}$

Jarosik et al. 2011, ApJS, 192, 14

H_0 [km s^{-1} Mpc^{-1}] . .	67.36 ± 0.54
Ω_Λ	0.6847 ± 0.0073
Ω_m	0.3153 ± 0.0073

Planck Collaboration, Aghanim et al. 2018, arXiv:1807.06209



Exam questions

1. Friedmann–Lemaître–Robertson–Walker metric
2. Friedmann equations and cosmological parameters

Literature

- P.J.E. Peebles, 1993, *Principles of physical cosmology*, Princeton University Press, Princeton, New Jersey, USA

Exercise 1

- You and an alien astronomer in a galaxy which for you is at $z = 1$ observe a quasar which for you is at $z = 2$ (along the same line of sight). What is the redshift of the quasar from the viewpoint of your alien colleague?

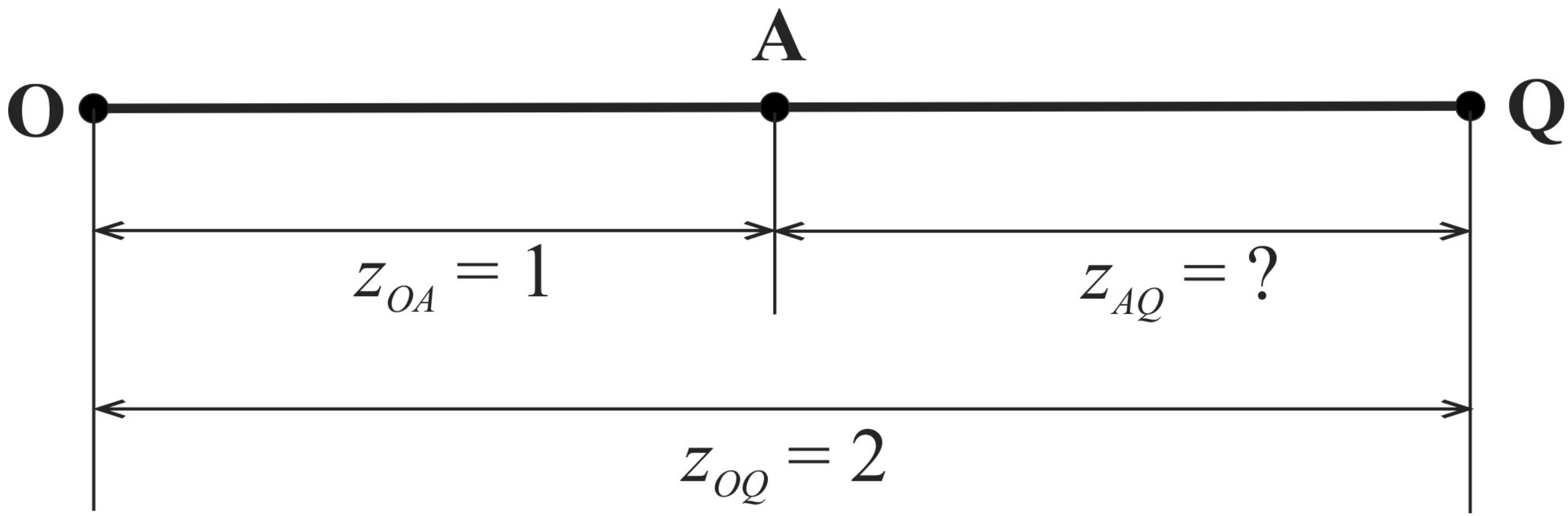
Exercise 2

- Assuming the flat Universe with the matter content which is predominantly non-relativistic ($p \approx 0$), find the constraints for Ω_M and Ω_Λ for which the Universe accelerates

Exercise 3

- Using the 1st Friedmann equation for ΛCDM cosmological model, derive the expression for age of spatially flat and matter dominated ($\Omega_\kappa = \Omega_\Lambda = 0$ and $\Omega_M = 1$) Einstein-de Sitter Universe

Solution 1



$$1 + z_{OQ} = \frac{a(t_O)}{a(t_Q)} = \frac{a(t_O)}{a(t_Q)}\frac{a(t_A)}{a(t_A)} = (1 + z_{OA})\,(1 + z_{AQ}) \Rightarrow$$

$$z_{AQ} = \frac{1 + z_{OQ}}{1 + z_{OA}} - 1 = \frac{1+2}{1+1} - 1 = \frac{1}{2}$$

Solution 2

- The acceleration Friedmann equation is: $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \quad (1)$$
- Definitions for Ω_M and Ω_Λ: $$\Omega_M = \frac{8\pi G\rho}{3H^2}, \quad \Omega_\Lambda = \frac{\Lambda c^2}{3H^2} \quad (2)$$
- By substituting (2) into (1) and for $p = 0$: $$\frac{\ddot{a}}{a} = H^2\left(\Omega_\Lambda - \frac{\Omega_M}{2}\right)$$
- The Universe is accelerating for $\Omega_\Lambda > \Omega_M / 2$, and decelerating for $\Omega_\Lambda < \Omega_M / 2$

Solution 3

1st Friedmann equation for $\Omega_\kappa = \Omega_\Lambda = 0$ and $\Omega_M = 1$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2}{a^3} \Rightarrow dt = \frac{1}{H_0}\sqrt{a}da \Rightarrow \int_0^{t_0} dt = \frac{1}{H_0}\int_0^1 \sqrt{a}da = \frac{2}{3H_0}$$