MASS 2023 Course:
Gravitation and Cosmology

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Lecture 10

- **Distance measures in cosmology:**
  - Proper distance
    - Hubble-Lemaître law
  - Comoving distance
  - Standard rulers and candles
    - Angular diameter distance
    - Luminosity distance
    - Etherington's distance-duality relation
  - Light travel (or lookback) time
  - Comoving volume and intersection probability
  - Exercises

Distance measures in cosmology

- Formula for conversion of cosmological redshift z into distance D?
- How to define D in expanding Universe?
- Answer: D may be defined in different ways, depending on how it is measured
- **FLRW metric:**

$$ds^2 = c^2 dt^2 - R^2(t)\left[d\chi^2 + S_k^2(\chi) d\Omega^2\right], \ r = S_k(\chi), \ S_k(\chi) = \begin{cases} \sin\chi, & k=+1 \\ \chi, & k=0 \\ \sinh\chi, & k=-1 \end{cases}$$

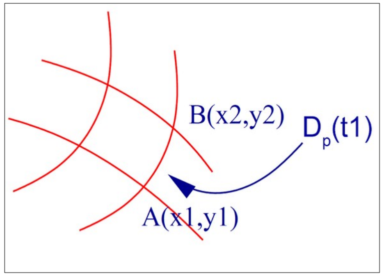
- r, χ - dimensionless comoving radial coordinates

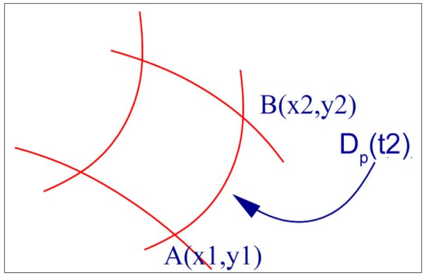
1. **Proper distance between two objects:** $D_P = R(t)\,\chi$
   - distance which would be measured with rulers at the time of observation
   - increasing with time
2. **Comoving distance between two objects:** $D_C = R_0\,\chi$
   - equals to proper distance times the relative scale factor of the Universe now to then, or $(1 + z)$:

   $$D_C = \frac{R_0}{R(t)} D_P = \frac{D_P}{a(t)} = (1+z) D_P$$

   - constant over time

- D_C is equal to D_P at the present time





Proper distance: Hubble-Lemaître law

- $R(t)$ represents the relative expansion of the Universe
- $R(t)$ relates the **proper distance** between a pair of objects, moving with the Hubble flow in an expanding or contracting Universe:

$$\boxed{D(t) = \frac{R(t)}{R(t_0)} D(t_0)}$$

- Lemaître, 1927, ASSB, 47, 49 (translated and reprinted in 1931, MNRAS, 91, 483):

$$\boxed{V_r = \dot{D} = \frac{\dot{R}}{R_0} D_0 = \frac{\dot{R}}{R} D = H \cdot D}$$

- z due to Doppler effect:

$$z = \frac{V_r}{c} \Rightarrow \boxed{D(z) = \frac{c}{H} z}$$

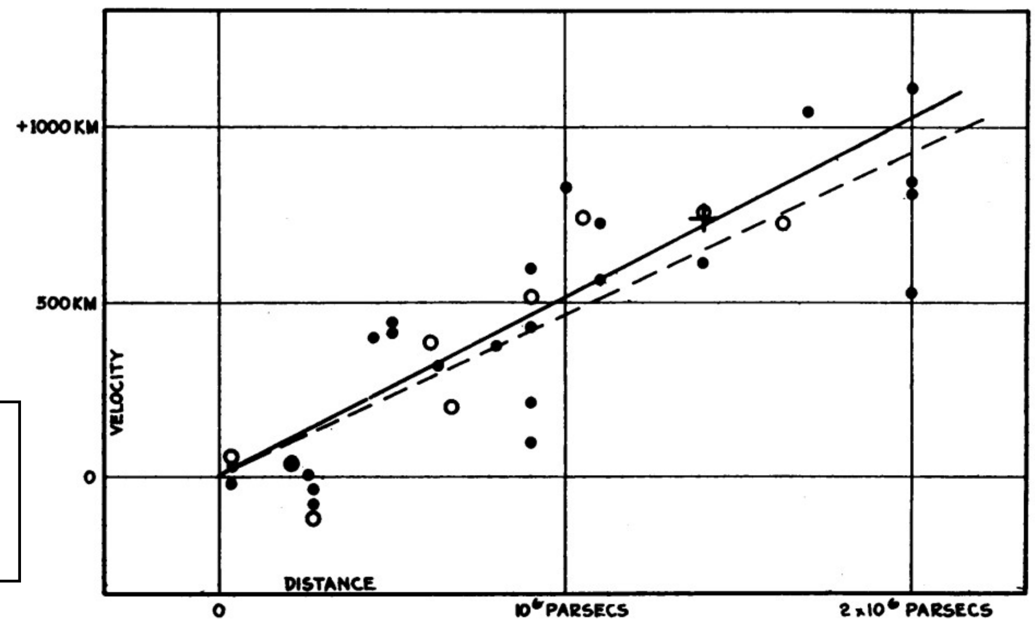


FIGURE 1

Velocity-Distance Relation among Extra-Galactic Nebulae.

Hubble, 1929, PNAS, 15, 168

Line-of-sight comoving distance

- Derivation from FLRW metric: $ds^2 = c^2 dt^2 - R^2(t)\left[d\chi^2 + S_k^2(\chi)d\Omega^2\right]$
- For photons: *ds = 0*, and for their radial tracks: $d\theta = d\varphi = 0$, so:

$$ds = 0 \wedge d\Omega = 0 \Rightarrow c\,dt = R(t)d\chi \Rightarrow d\chi = c\frac{dt}{R(t)} \Rightarrow \boxed{\chi(t) = c\int_t^{t_0} \frac{dt'}{R(t')}}$$

- On the other hand:

$$1 + z = \frac{R_0}{R} \Rightarrow \dot{z} = -R_0\frac{\dot{R}}{R^2} \Leftrightarrow \frac{dz}{dt} = -\frac{R_0}{R}H \Leftrightarrow \boxed{\frac{R_0}{R(t)}dt = -\frac{dz}{H(z)}}$$

$$D_C(t) = R_0\chi(t) = c\int_t^{t_0} \frac{R_0}{R(t')}dt' \Leftrightarrow \boxed{D_C(z) = c\int_0^z \frac{dz'}{H(z')}}$$

- *H*(*z*) is given by the 1st Friedmann equation for ΛCDM model, and then:
- **<u>Line-of-sight comoving distance</u>:**

$$\boxed{D_C(z) = D_H \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_\kappa(1+z')^2 + \Omega_\Lambda}},} \qquad D_H = \frac{c}{H_0}$$

- Theoretical cosmological distance along radial coordinate in FLRW metric

Transverse comoving distance

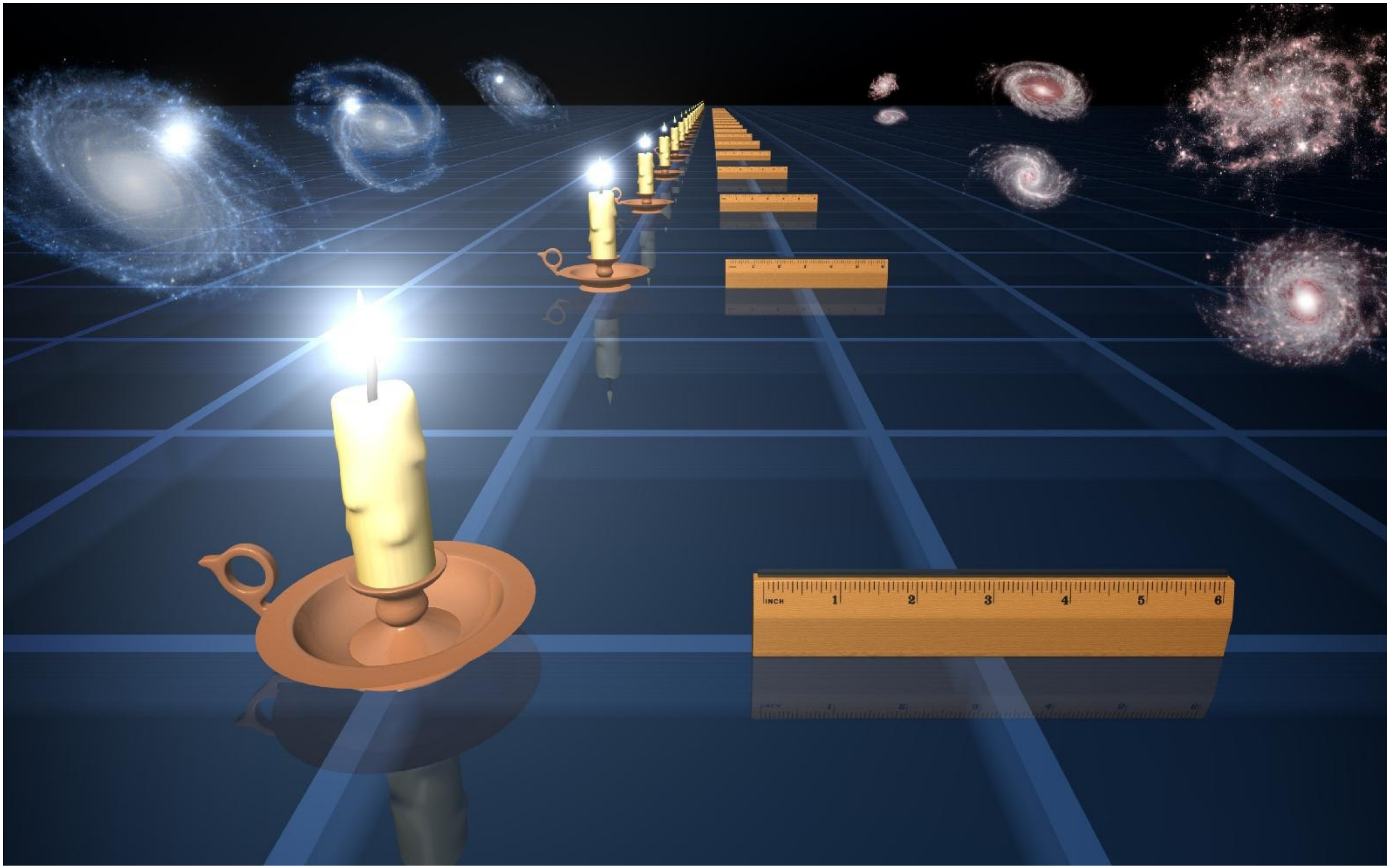
- If $\delta\theta$ is an angle between two objects at the same z, then the distance between them is: $\delta\theta\, D_{\mathrm{M}}(z)$, where $D_{\mathrm{M}}(z)$ is their **transverse comoving distance:**

$$D_M(z) = \begin{cases} D_H \frac{1}{\sqrt{\Omega_\kappa}} \sinh[\sqrt{\Omega_\kappa} D_C(z)/D_H], & \Omega_\kappa > 0 \\ D_C(z), & \Omega_\kappa = 0 \\ D_H \frac{1}{\sqrt{|\Omega_\kappa|}} \sin[\sqrt{|\Omega_\kappa|} D_C(z)/D_H], & \Omega_\kappa < 0 \end{cases}$$

- Analytic solution:

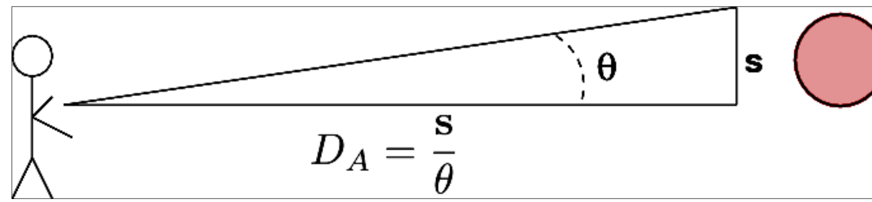
$$D_M(z) = D_H \frac{2\left[2 - \Omega_M(1-z) - (2-\Omega_M)\sqrt{1+\Omega_M z}\right]}{\Omega_M^2(1+z)},\ \Omega_\Lambda = 0$$

Measuring distances by standard rulers and standard candles



Angular diameter distance

- D_A is Measured by standard cosmological rulers: objects which linear diameter $\boldsymbol{s}$ is known and does not change with cosmological time
- Defined as the ratio of an object's physical transverse size $\boldsymbol{s}$ to its observed angular size θ (in radians)



- **Calculating D_A from z:**

$$\boxed{D_A(z) = \frac{D_M(z)}{1+z}}$$

- Angular diameter distance between two objects at redshifts z_1 and z_2:

$$D_{A12} = \frac{1}{1+z_2}\left[D_{M2}\sqrt{1+\Omega_\kappa\frac{D_{M1}^2}{D_H^2}} - D_{M1}\sqrt{1+\Omega_\kappa\frac{D_{M2}^2}{D_H^2}}\right]$$

where D_{M1} and D_{M2} are the corresponding comoving distances

- D_A is often used in the theory of gravitational lensing

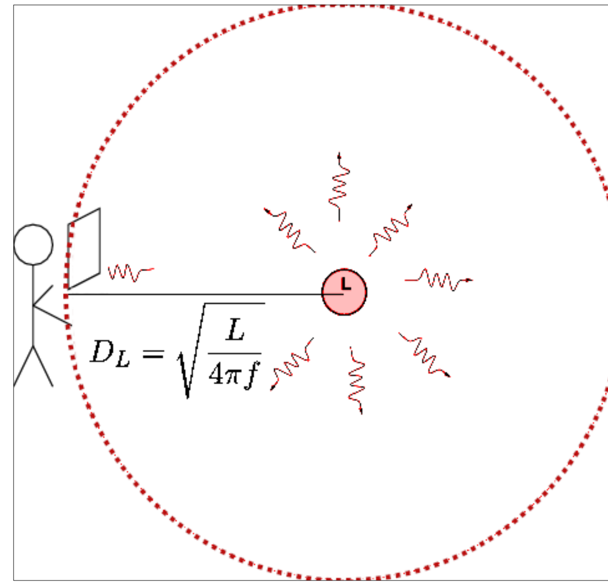
Luminosity distance

- D_L is obtained from observed flux f of standard cosmological candles: objects which luminosity L is known and does not change with cosmological time
- D_L is used in the case of SN Ia
- **Calculating D_L from z:**

$$\boxed{D_L(z) = (1+z)D_M(z)}$$

- **Etherington's distance-duality relation:**

$$D_L(z) = (1+z)^2 D_A(z)$$



- Valid for all cosmological models based on the Riemannian geometry
- Using inappropriate D causes errors which increase with z
- Typical mistake: calculating L from f using proper distance obtained by Hubble's law (instead of D_L)

Light-travel distance

- **Light-travel (lookback) time**: time that it took light to reach the observer
- Difference between the age of the universe at the time of observation and its age at the time when the light was emitted:

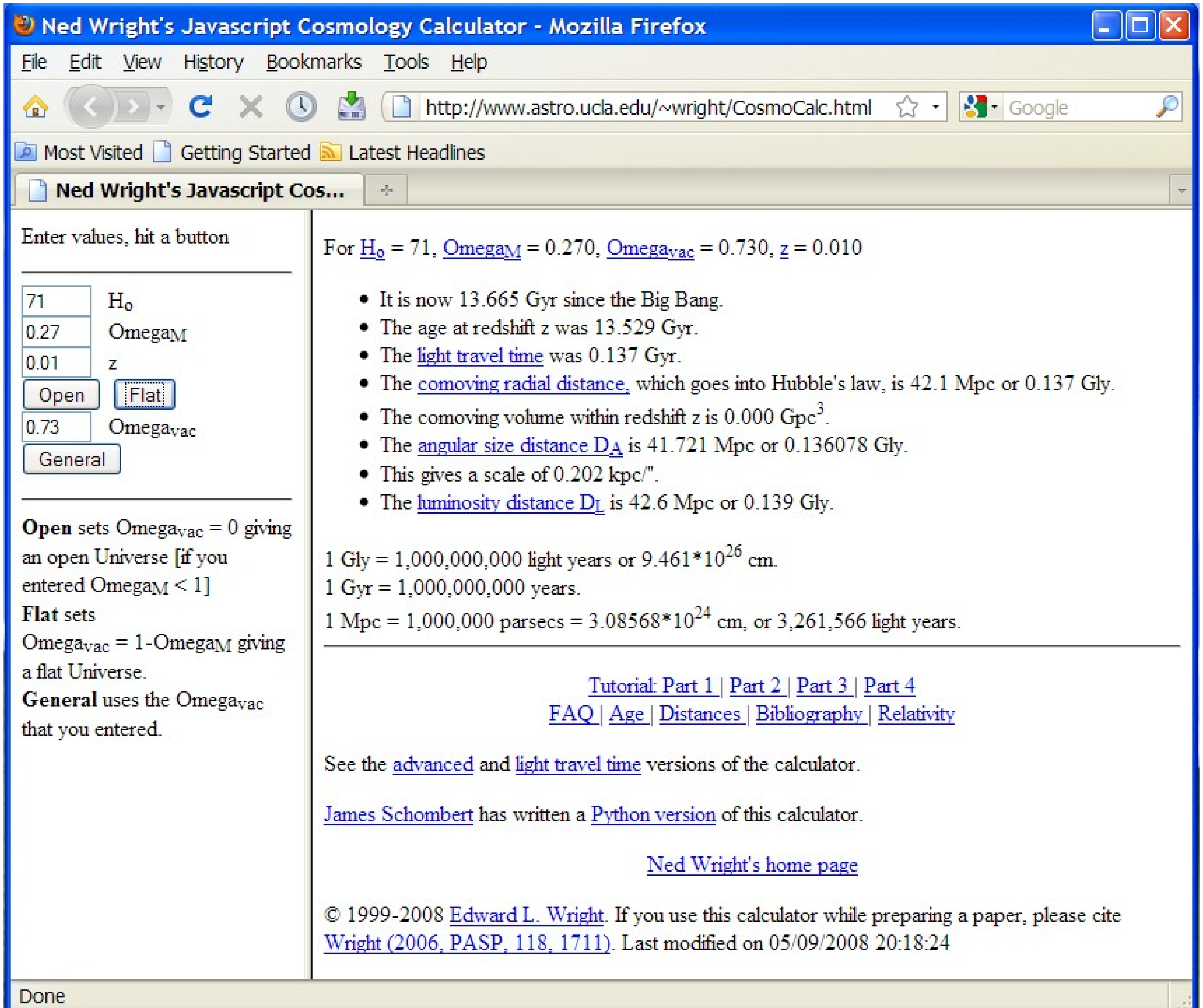
$$\boxed{t_L(z) = t_H \int_0^z \frac{dz'}{(1+z')E(z')}}$$

where $E(z) = \dfrac{H(z)}{H_0} = \sqrt{\Omega_M (1+z')^3 + \Omega_\kappa (1+z')^2 + \Omega_\Lambda}$

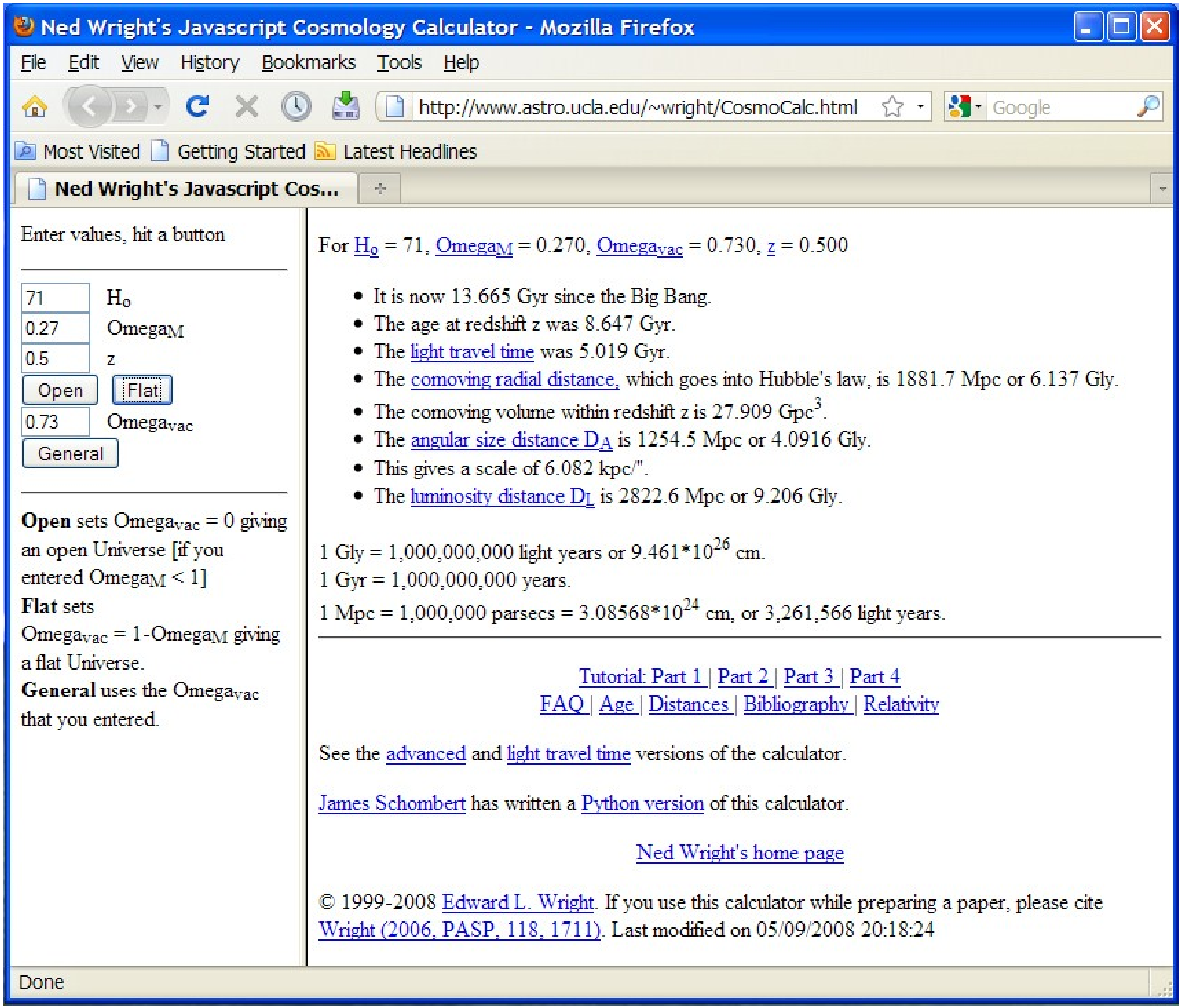
- Corresponding distance is: $\boxed{D_T(z) = c \cdot t_L(z)}$

- t_L is used for time delays between images of gravitational lenses

Numerical comparison of cosmological distances I



Numerical comparison of cosmological distances II



Ned Wright's Javascript Cosmology Calculator - Mozilla Firefox

File Edit View History Bookmarks Tools Help

http://www.astro.ucla.edu/~wright/CosmoCalc.html

Most Visited Getting Started Latest Headlines

Ned Wright's Javascript Cos...

Enter values, hit a button

71 H_o

0.27 Omega_M

0.5 z

Open Flat

0.73 Omega_{vac}

General

Open sets $\text{Omega}_{vac} = 0$ giving an open Universe [if you entered $\text{Omega}_M < 1$]

Flat sets $\text{Omega}_{vac} = 1\text{-Omega}_M$ giving a flat Universe.

General uses the Omega_{vac} that you entered.

For $H_o = 71$, $\text{Omega}_M = 0.270$, $\text{Omega}_{vac} = 0.730$, $z = 0.500$

- It is now 13.665 Gyr since the Big Bang.
- The age at redshift z was 8.647 Gyr.
- The light travel time was 5.019 Gyr.
- The comoving radial distance, which goes into Hubble's law, is 1881.7 Mpc or 6.137 Gly.
- The comoving volume within redshift z is 27.909 Gpc^3.
- The angular size distance D_A is 1254.5 Mpc or 4.0916 Gly.
- This gives a scale of 6.082 kpc/".
- The luminosity distance D_L is 2822.6 Mpc or 9.206 Gly.

1 Gly = 1,000,000,000 light years or 9.461×10^{26} cm.

1 Gyr = 1,000,000,000 years.

1 Mpc = 1,000,000 parsecs = 3.08568×10^{24} cm, or 3,261,566 light years.

Tutorial: Part 1 | Part 2 | Part 3 | Part 4
FAQ | Age | Distances | Bibliography | Relativity

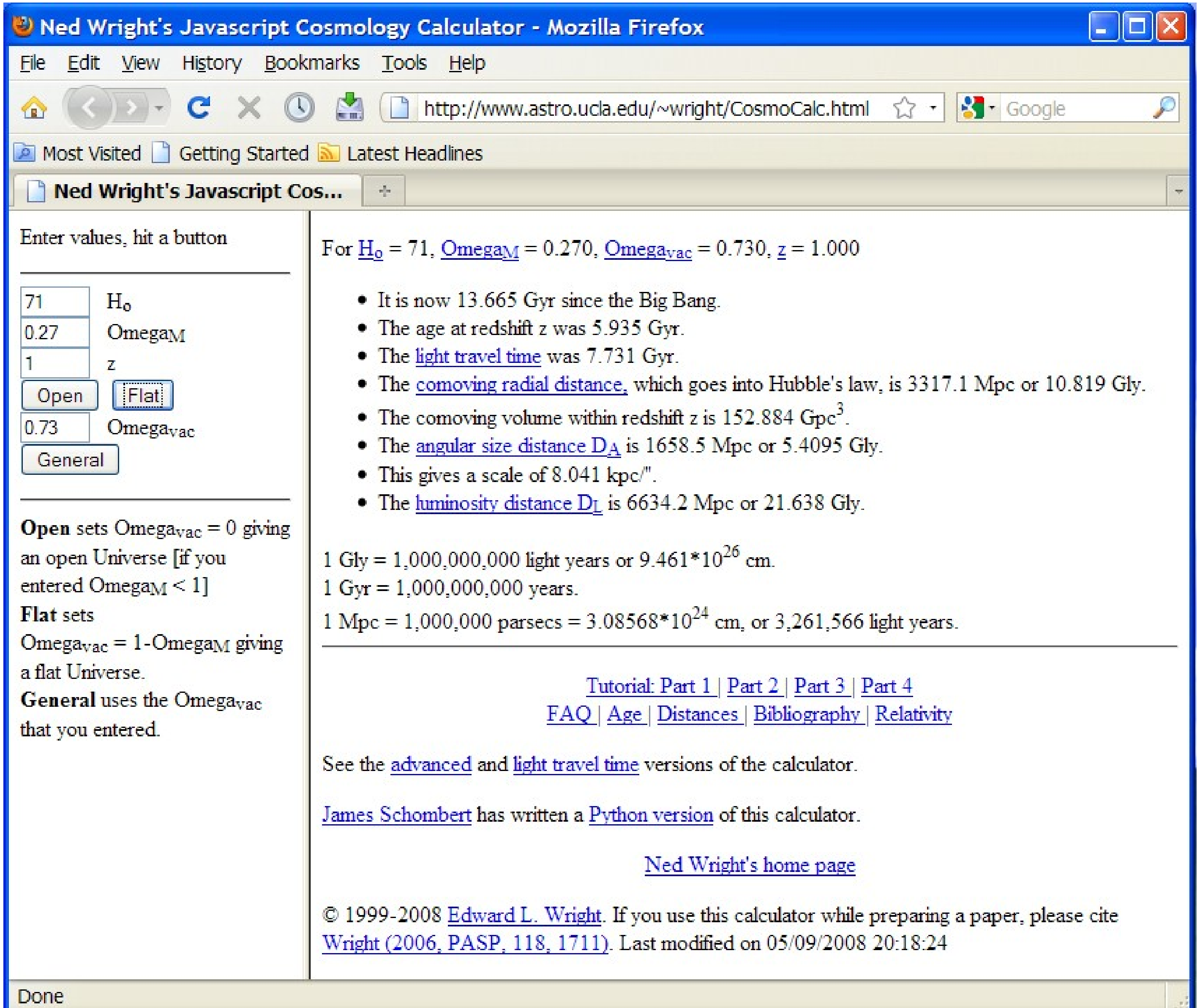
See the advanced and light travel time versions of the calculator.

James Schombert has written a Python version of this calculator.

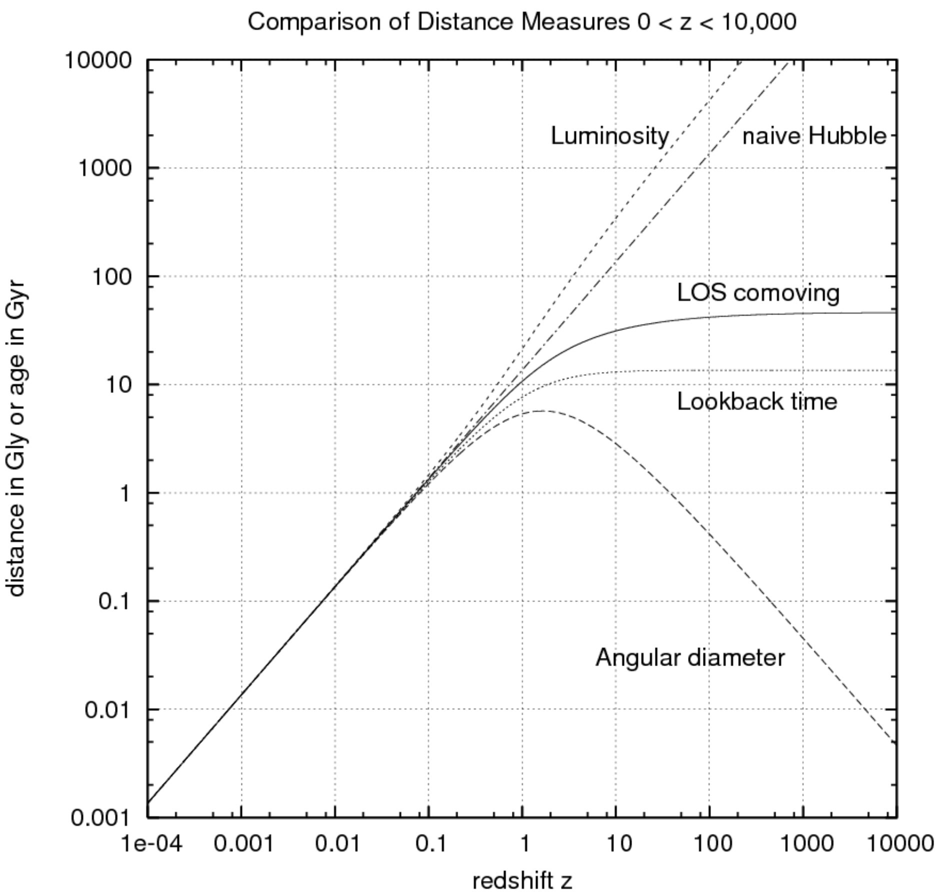
Ned Wright's home page

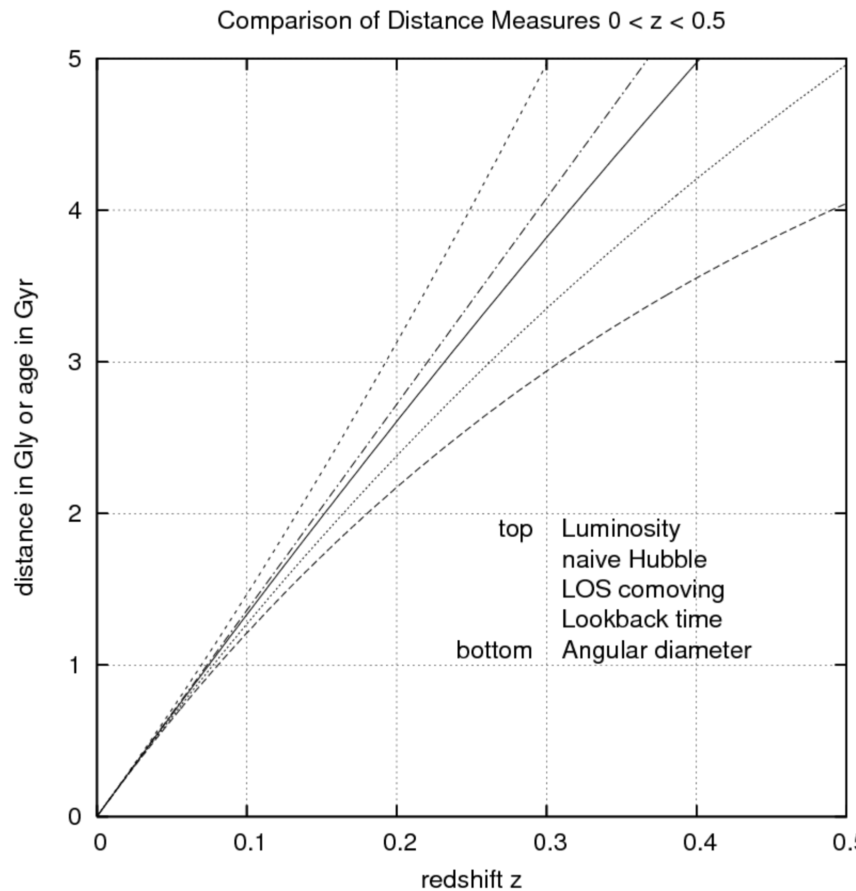
Done

Numerical comparison of cosmological distances III



Graphical comparison between cosmological distances





$H_0 = 72$ km/s/Mpc, $\Omega_M = 0.27$, $\Omega_\Lambda = 0.73$

Comoving volume

- Volume in which number density of non-evolving objects locked into Hubble flow is constant with z
- Equals to proper volume times three factors of the relative scale factor now to then, or $(1 + z)^3$
- Element of comoving volume for solid angle $d\Omega$ and redshift interval dz:

$$\boxed{dV_C(z) = D_H \frac{(1+z)^2 D_A^2(z)}{E(z)} d\Omega dz}$$

- Total all-sky comoving volume, out to redshift z:

$$V_C = \begin{cases} \left(\frac{4\pi D_H^3}{2\Omega_\kappa}\right)\left[\frac{D_M}{D_H}\sqrt{1+\Omega_\kappa \frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_\kappa|}} \operatorname{arcsinh}\left(\sqrt{|\Omega_\kappa|}\frac{D_M}{D_H}\right)\right], & \Omega_\kappa > 0 \\ \frac{4\pi}{3} D_M^3, & \Omega_\kappa = 0 \\ \left(\frac{4\pi D_H^3}{2\Omega_\kappa}\right)\left[\frac{D_M}{D_H}\sqrt{1+\Omega_\kappa \frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_\kappa|}} \arcsin\left(\sqrt{|\Omega_\kappa|}\frac{D_M}{D_H}\right)\right], & \Omega_\kappa < 0 \end{cases}$$

Intersection probability

- Given a population of objects with:

  1. $n(z)$ - comoving number density (number per unit volume) and

  2. $\sigma(z)$ - cross section (area)

- Differential probability that a line of sight will intersect one of the objects in redshift interval dz at redshift z:

$$\boxed{dP(z) = n(z)\sigma(z) D_H \frac{(1+z)^2}{E(z)} dz}$$

Exam question

1. Distance measures in cosmology

Literature

Articles:

1. David W. Hogg, 2000, *Distance measures in cosmology*, arXiv:astro-ph/9905116v4
2. Carroll, S. M., Press, W. H. & Turner, E. L. 1992, *The cosmological constant*, ARA&A, 30, 499
3. Davis, T. M. & Lineweaver, C. H. 2004, *Expanding Confusion: Common Misconceptions of Cosmological Horizons and the Superluminal Expansion of the Universe*, PASA, 21, 97 (arXiv:astro-ph/0310808)

Online cosmology calculators:

4. A list of cosmology calculators: http://ned.ipac.caltech.edu/help/cosmology_calc.html
5. Ned Wright's Javascript Cosmology Calculator: http://www.astro.ucla.edu/~wright/CosmoCalc.html

Exercise 1

Consider two flat ($\Omega_k = 0$), expanding universes: One that is matter dominated, i.e. $\Omega_{\mathrm{M}} = 1$, and one that is dominated by a cosmological constant, i.e. $\Omega_\Lambda = 1$.

a) Derive for both universes an expression for the comoving distance as a function of redshift z.

b) The "observable universe" is the part of the universe up to redshift $z = \infty$. Compute for both of our universes the comoving volume of this region. Is there a qualitative difference between the two volumes?

Exercise 2

Assume a spatially flat universe, with $\Omega_{\mathrm{tot}} = \Omega_M = 1$ and derive the formulas for angular diameter distance $D_A(z)$ and luminosity distance $D_L(z)$ in this cosmological model.

Exercise 3

Assume a flat ($\Omega_k = 0$) cosmological model with $H_0 = 71$/km/s/Mpc and $\Omega_M = 0.27$, and graphically compare the following cosmological distances: $D_C(z)$, $D_A(z)$, $D_L(z)$ and naive Hubble distance: $D_H(z) = c\, z \,/\, H_0$ (see Hogg 2000 and Davis & Lineweaver 2004) .

Exercise 4

Write a Python script which computes angular diameter distance between two objects at redshifts z_1 and z_2 in the case of a flat ($\Omega_k = 0$) cosmological model with $H_0 = 71$/km/s/Mpc and $\Omega_M = 0.27$, and use it to reconsider the exercise with an alien astronomer by calculating all three angular diameter distances: $D_A(0, z_a)$ - between you and an alien at redshift $z_a = 1$, $D_A(0, z_q)$ - between you and a quasar at $z_q = 2$ and $D_A(z_a, z_q)$ - between alien and quasar.

Solution 1

a) $$D_c(z) = D_H \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_\kappa(1+z')^2 + \Omega_\Lambda}}$$

1. $\Omega_M = 1, \Omega_\Lambda = 0, \Omega_\kappa = 0$: $D_C(z) = D_H \int_0^z \frac{dz'}{\sqrt{(1+z')^3}} = 2D_H\left(1 - \frac{1}{\sqrt{1+z}}\right)$

2. $\Omega_M = 0, \Omega_\Lambda = 1, \Omega_\kappa = 0$: $D_C(z) = D_H \int_0^z dz' = D_H z$

b) $\Omega_\kappa = 0$: $V_C(z) = \frac{4\pi}{3} D_M^3(z) = \frac{4\pi}{3} D_C^3(z)$

1. $\Omega_M = 1, \Omega_\Lambda = 0$: $V_C = \frac{32\pi}{3} D_H^3$

2. $\Omega_M = 0, \Omega_\Lambda = 1$: $V_C = \infty$

- The volume of the observable Universe in the case of the cosmological constant is not finite, while it is finite in the case of a matter dominated universe

Solution 2

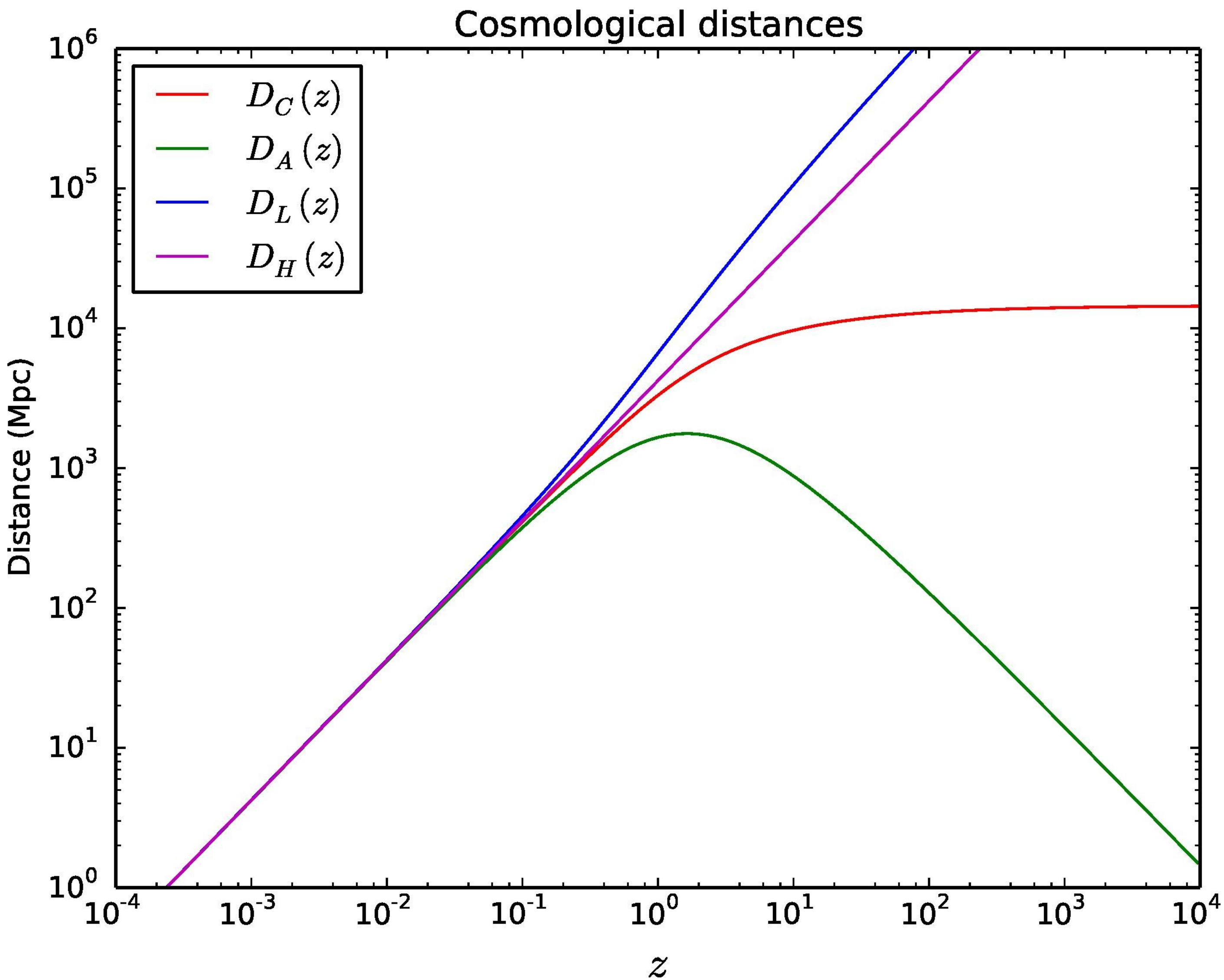
$$\Omega_\kappa = 0 \Rightarrow D_A(z) = \frac{D_M(z)}{1+z} = \frac{D_C(z)}{1+z}$$

$$\Omega_M = 1 \wedge \Omega_\Lambda = 0 \wedge \Omega_\kappa = 0 \Rightarrow D_C(z) = 2D_H \left(1 - \frac{1}{\sqrt{1+z}}\right) \Rightarrow$$

$$D_A(z) = \frac{2D_H}{1+z}\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

$$D_L(z) = (1+z)^2 D_A(z) = 2D_H(1+z)\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

Solution 3



Solution is obtained by Python script in "cosm_dist.py"

Solution 4

Output from Python script in "DA.py":

Enter the 1st redshift: 1.0
Enter the 2nd redshift: 2.0
Enter the Hubble constant: 71.0
Enter the Omega_matter: 0.27

Angular diameter distance DA(0,z1) = 1658.7 Mpc
Angular diameter distance DA(0,z2) = 1748.4 Mpc
Angular diameter distance DA(z1,z2) = 642.6 Mpc