MASS 2023 Course:
Gravitation and Cosmology

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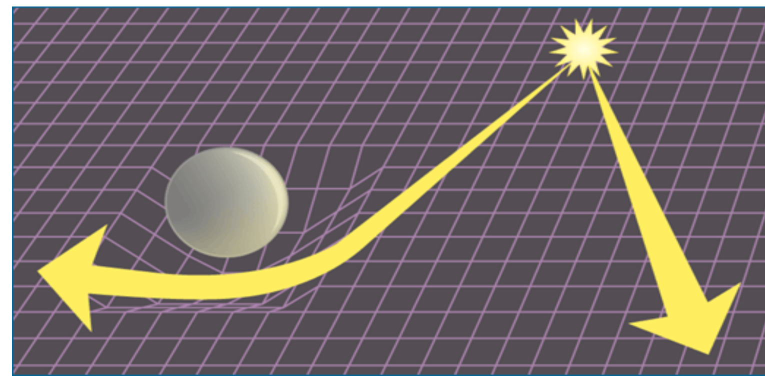
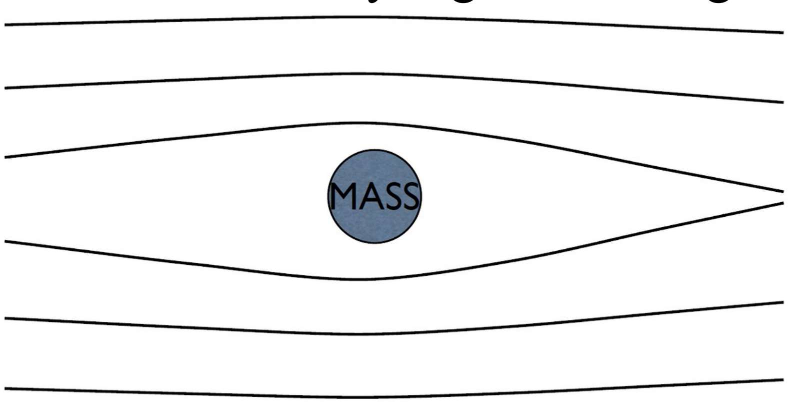
Lecture 12: gravitational lenses and their cosmological applications

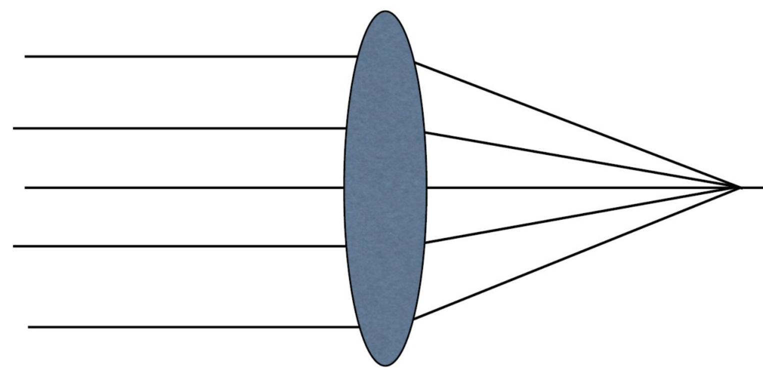
- Gravitational lenses:
  - Definition and types: **strong** (macro and micro) and **weak** lensing
  - Point-like lenses: deflection angle, lens equation and Einstein radius
  - Extended lenses: surface mass density and deflection (lensing) potential
- Cosmological applications of gravitational lensing
  - Strong lenses as natural telescopes: detection of the most distant galaxies
  - Fermat potential and lensing time delay: determination of H_0
  - Optical depth and statistics of strong lenses: constraints on the cosmological parameters
  - Shape distortions due to weak lensing and mass reconstruction: detection of dark matter
- Problems with standard ΛCDM cosmological model: Hubble tension and cosmological constant problem (vacuum catastrophe)
- Exercises

Light bending in the gravitational field

- **Gravitational lens** is a massive celestial object (or a distribution of matter), located between an observer and a distant background source, which gravitational force deflects the light rays from the source, producing the multiple images of the source (macrolensing), amplification of its brightness (microlensing), or distortion of its shape (weak lensing).
- Light "falls" under the gravity
- Newton: gravitational bending of the paths of "corpuscles" (particles from which light is composed)
- General Relativity: light follows geodesics



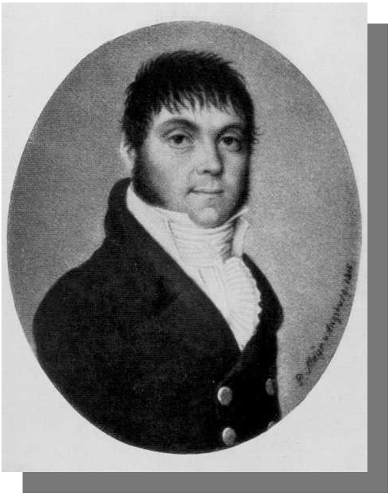


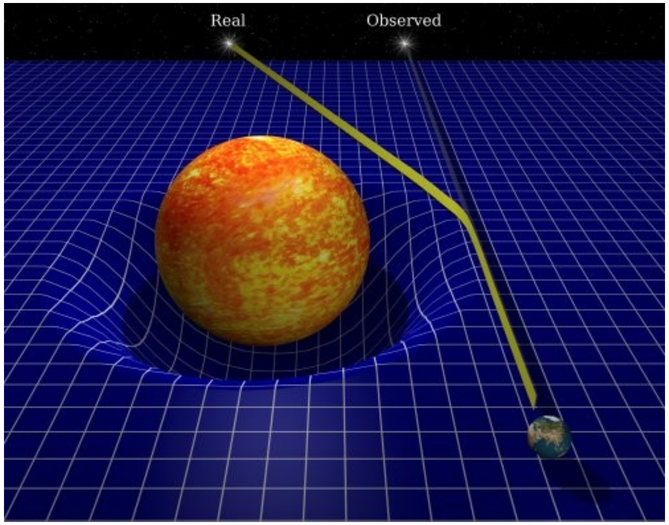


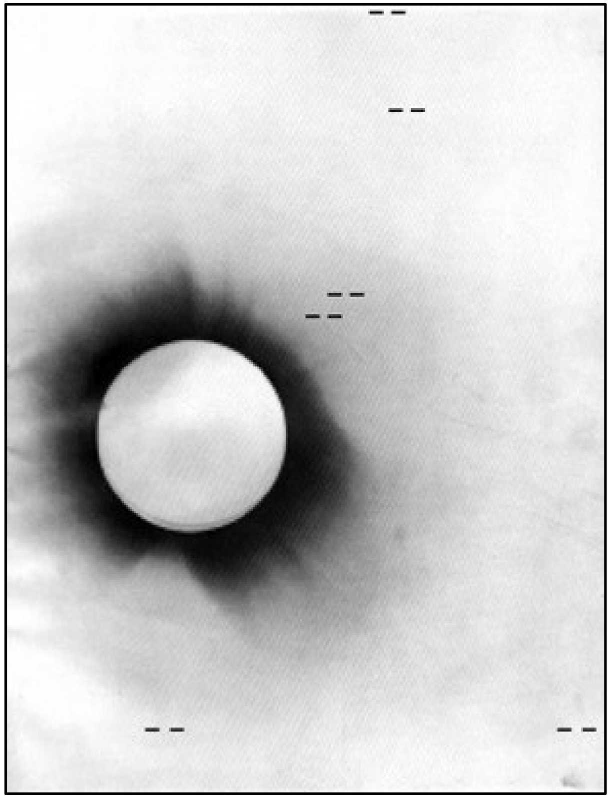
Gravitational lens:
1. deflection is larger closer to the axis
2. focusing to a line

Converging lens in optics:
1. deflection is larger further from the axis
2. focusing to a point

Light deflection angle





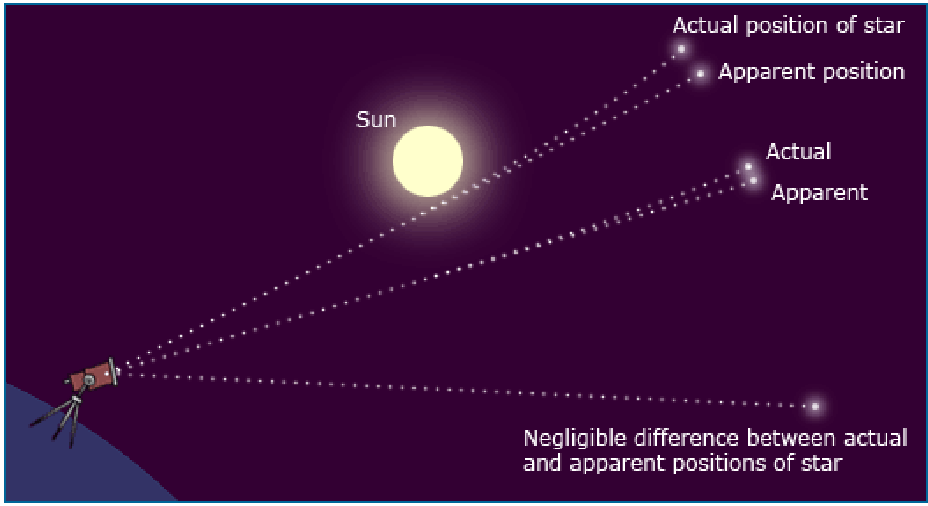


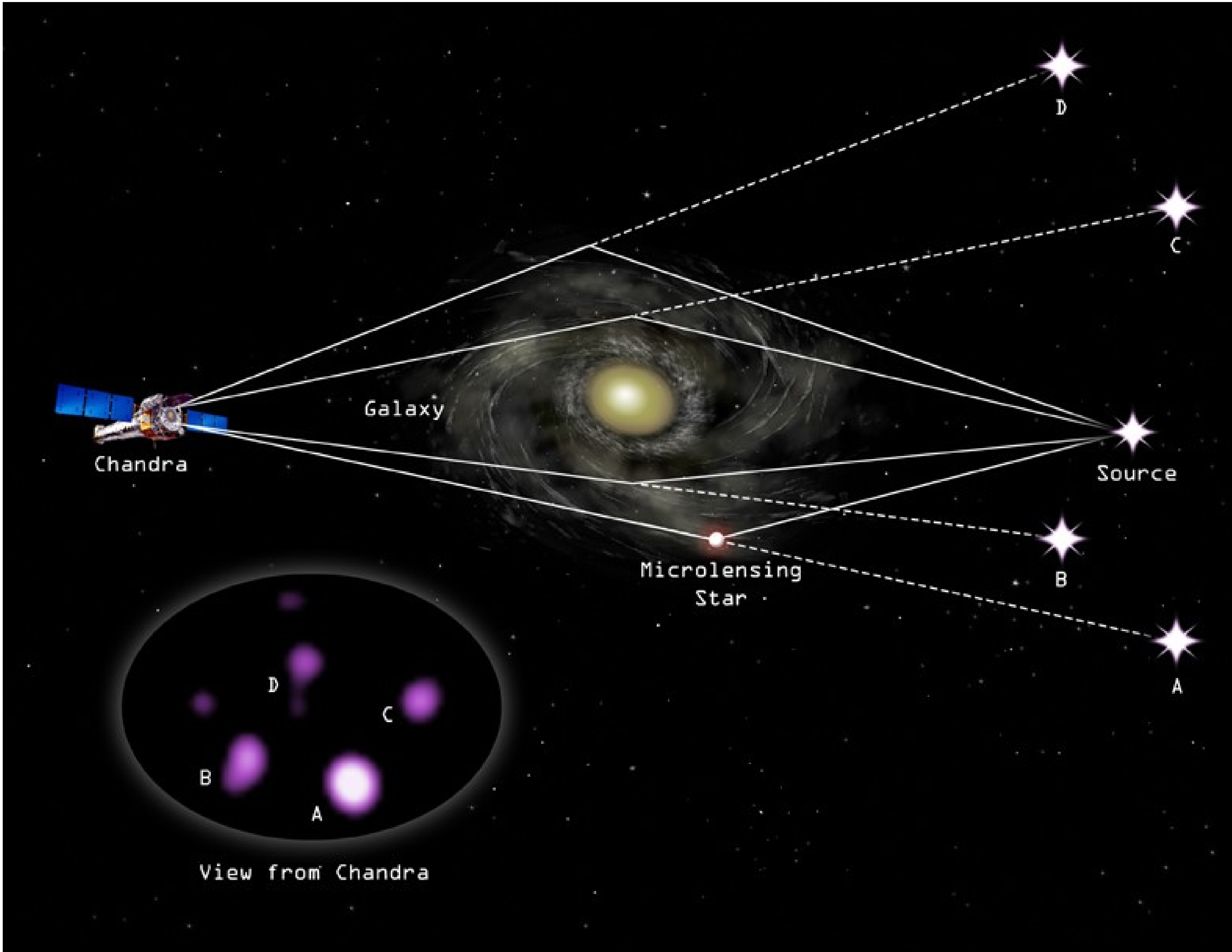
- Johann Georg von Soldner (1804) - trajectory of particle with speed *c*: $\alpha = \frac{2GM}{c^2\xi}$
- Albert Einstein (1915) - General Relativity:

$$\boxed{\alpha = \frac{4GM}{c^2\xi}}$$

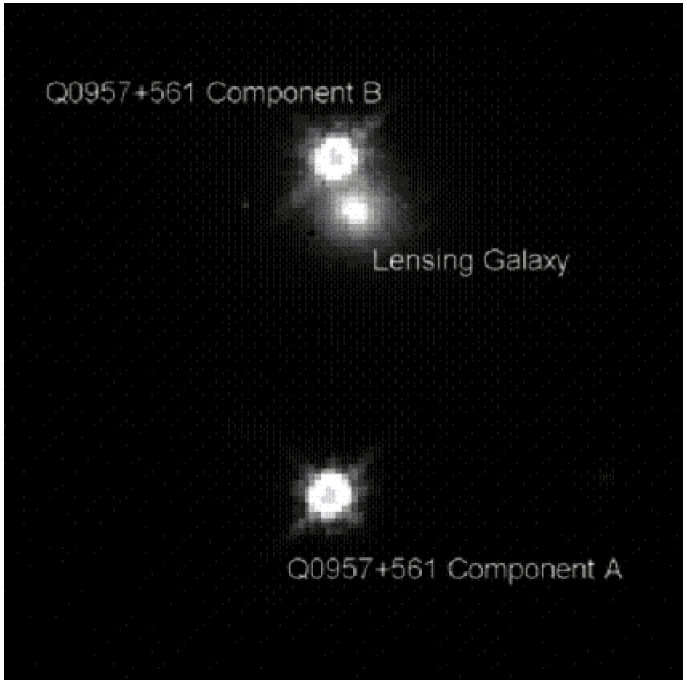
- Eddington - total solar eclipse in 1919:
  - No light bending: $\alpha = 0''$
  - Newton's mechanics: $\alpha = 0''.87$
  - GR: $\alpha = 1''.75$
- Confirmation of Einstein's predictions:

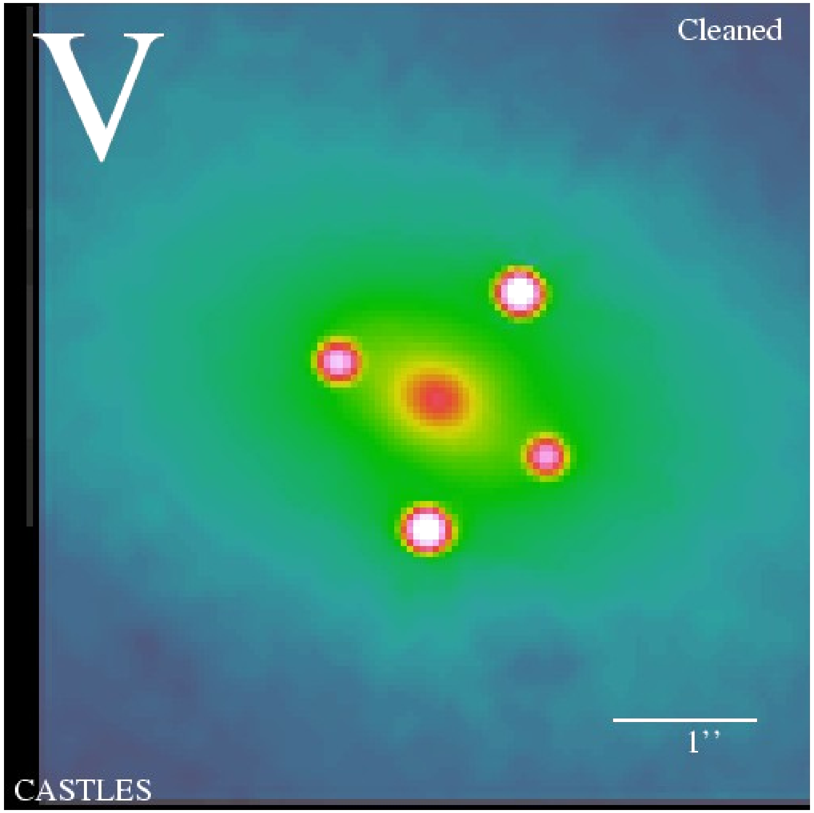
$\alpha_1 = 1''.98 \pm 0''.12 \qquad \alpha_2 = 1''.61 \pm 0''.30$



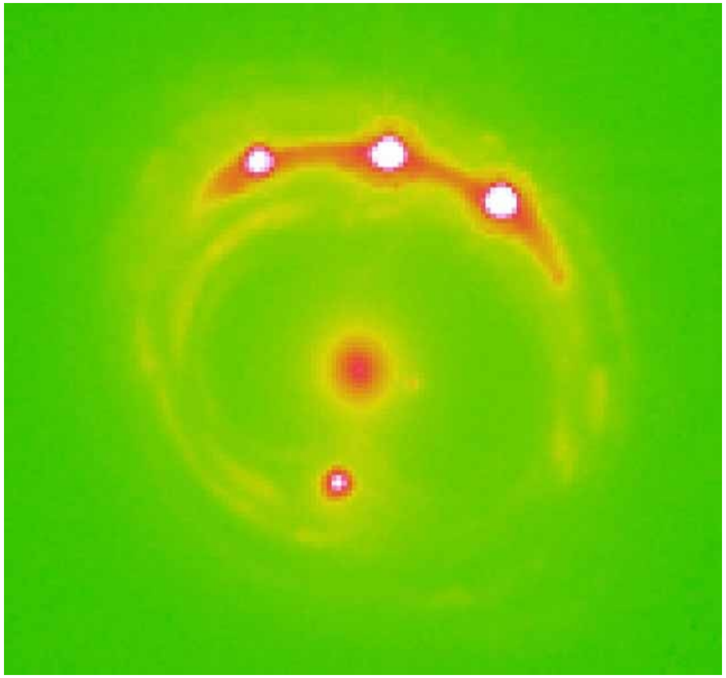

D
C
Galaxy
Chandra
Source
B
Microlensing
Star
A
D
C
B
A
View from Chandra

- The first identified gravitational lens: double-imaged quasar QSO 0957+561 (right), discovered by Dennis Walsh, Robert Carswell and Ray Weyman in 1979
- Both components have identical redshifts and spectra





Quasar Q2237+030 at z=1.695 (Einstein cross) and lensing galaxy ZW2237+030 at z=0.0394



Quasar RXJ1131-1231

Gravitational lensing theory

- **Geometrically thin lens:** the field equations of GR can be linearized if the gravitational field is weak (i.e. for the small deflection angle), and the ray can be approximated as a straight line near the deflecting mass
- **Lens equation** (see the figure):

$$\vec{\eta} = \frac{D_s}{D_d}\vec{\xi} - D_{ds}\vec{\hat{\alpha}}(\vec{\xi})$$

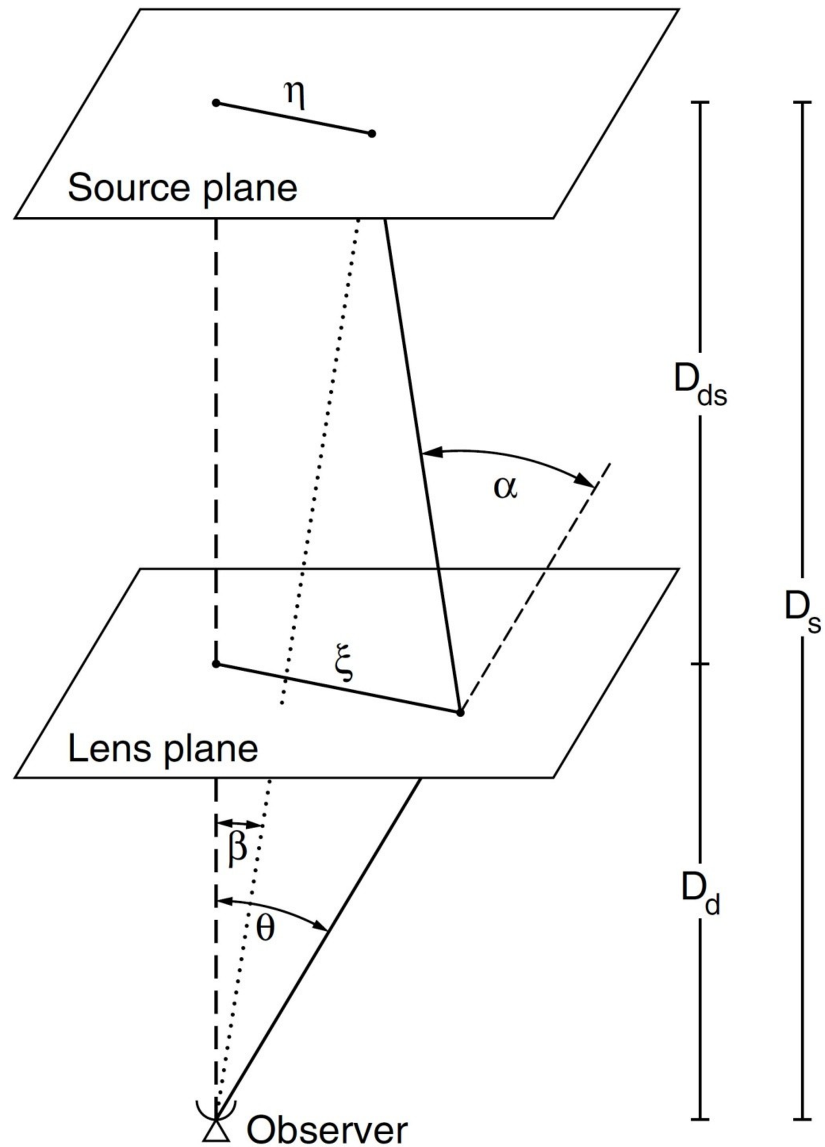
- **Light deflection angle**: $\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4GM}{c^2}\frac{\vec{\xi}}{\xi^2}$
- Angular coordinates $\vec{\beta}$ and $\vec{\theta}$, and **scaled (reduced) deflection angle** $\vec{\alpha}(\vec{\theta})$:

$$\vec{\eta} = D_s\vec{\beta},\ \vec{\xi} = D_d\vec{\theta},\ \vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s}\vec{\hat{\alpha}}(\vec{\theta}) \Rightarrow$$

- Dimensionless lens equation:

$$\boxed{\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})}$$

- Angular diameter distances between the observer and lens, observer and source, and lens and source: $D_d = D_A(0, z_d),\ D_s = D_A(0, z_s),\ D_{ds} = D_A(z_d, z_s)$

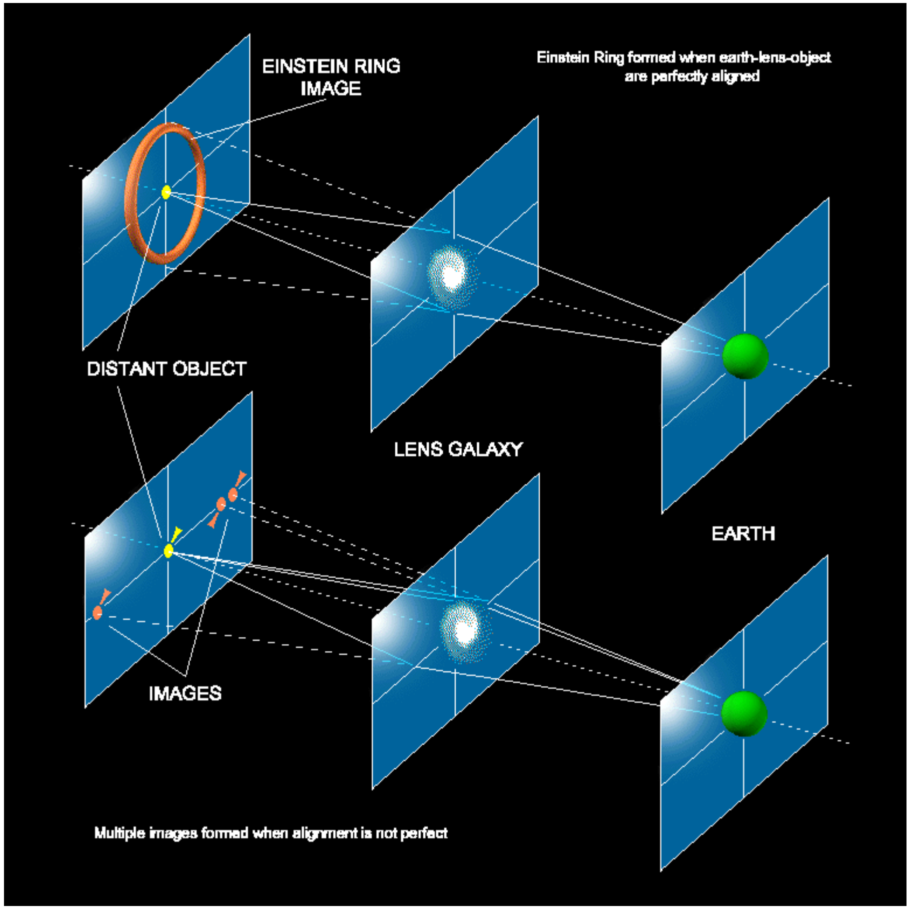


Einstein radius

- Solution of the lens equation for a point mass M and perfect alignment between the observer, lens and source: $\vec{\eta} = \vec{\beta} = 0 \Rightarrow$ **Einstein radius**
- **Linear** (in the lens plane): $\xi_E = \sqrt{\frac{4GM}{c^2}\frac{D_d D_{ds}}{D_s}}$
- **Angular:** $\boxed{\theta_E = \frac{\xi_E}{D_d} = \sqrt{\frac{4GM}{c^2 D}}}$, where

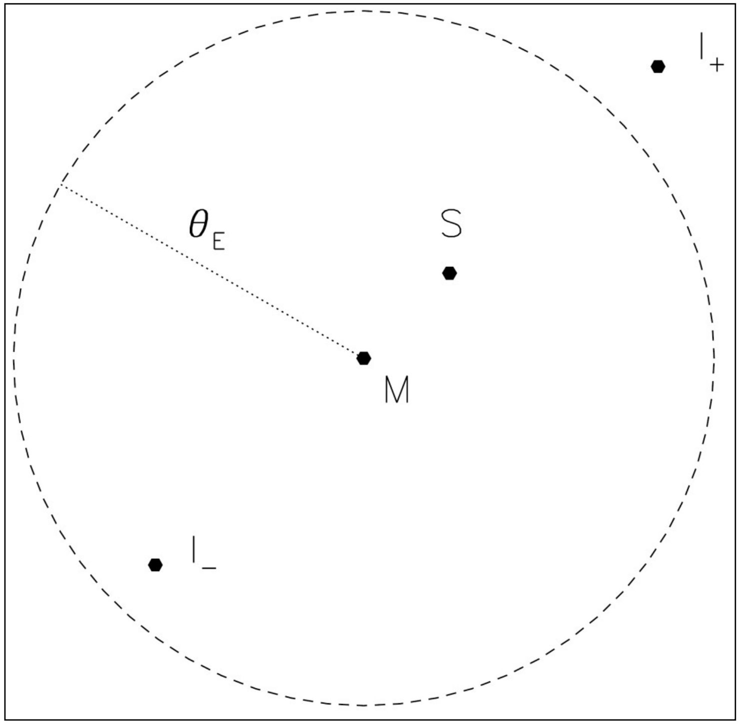
  D is **effective lens distance:** $D = \frac{D_d D_s}{D_{ds}}$
- **Projected** (in the source plane):

  $$\eta_E = \frac{D_s}{D_d}\xi_E = \sqrt{\frac{4GM}{c^2}\frac{D_s D_{ds}}{D_d}}$$
- Typical Einstein radius:
  - for a galaxy: on the order of 1"
  - for galaxy clusters: on the order of 10"
  - for a star: on the order of μas
- Separation between the images is twice the average Einstein radius
- Powerful method for measuring the masses of distant objects



Point-like lenses

- Lens and source positions normalized to θ_E:
$$\vec{x} = \frac{\vec{\theta}}{\theta_E} \wedge \vec{y} = \frac{\vec{\beta}}{\theta_E} \Rightarrow$$
- Lens equation: $\vec{y} = \vec{x} - \frac{\vec{x}}{x^2} \Leftrightarrow \vec{y} = \vec{x} - \frac{\vec{e}}{x} \Rightarrow$
- **Image positions**: $$\boxed{x_{1,2} = \frac{1}{2}\left(y \pm \sqrt{y^2+4}\right)}$$
- **Magnification of an image** - ratio between the solid angles of the image and the source, and for a circularly symmetric lens: $\mu = \frac{\theta}{\beta}\frac{d\theta}{d\beta}$
- In the case of a point lens:

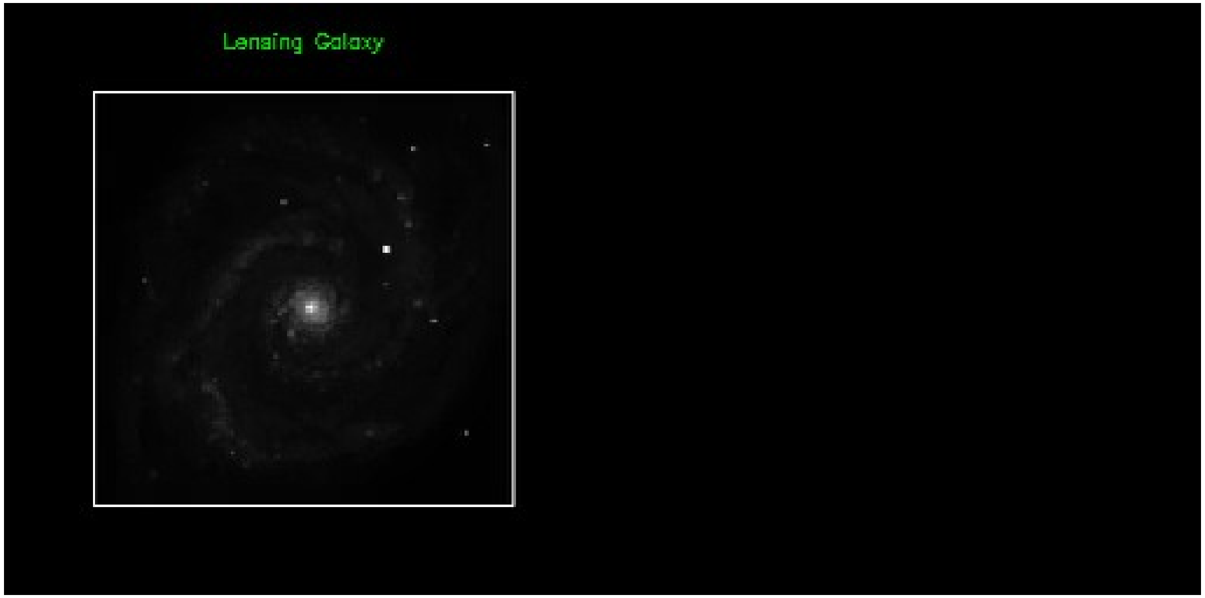


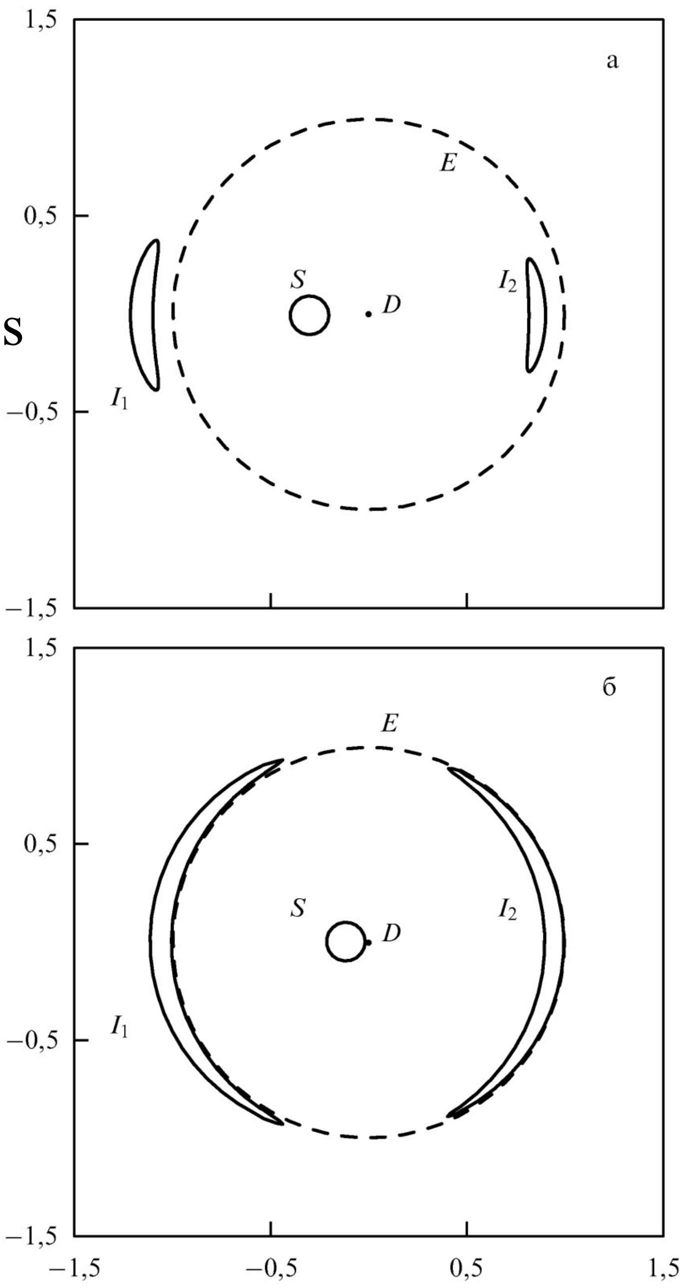
$$\boxed{\mu_{1,2} = \left(1 - \left[\frac{\theta_E}{\theta_{1,2}}\right]^4\right)^{-1} \Leftrightarrow \mu_{1,2} = \left(1 - \frac{1}{x_{1,2}^4}\right)^{-1} \Leftrightarrow \mu_{1,2} = \frac{1}{2} \pm \frac{y^2+2}{2y\sqrt{y^2+4}}}$$

- Magnification of the image inside the Einstein radius is negative
- $y = 0 \Rightarrow$ Einstein ring of a point source has infinite magnification
- **Total magnification**: $\mu = |\mu_1| + |\mu_2| = \mu_1 - \mu_2 = \frac{y^2+2}{y\sqrt{y^2+4}} > 1$

Images of extended sources

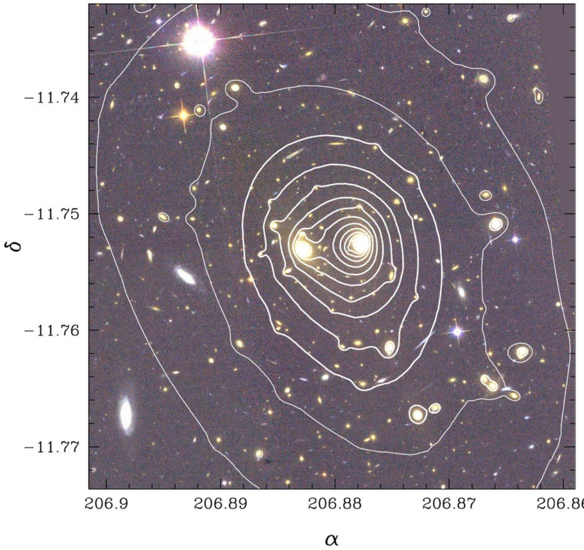
- Point-like lenses produce 2 mirror-inverted images





Images of a circular (right) and an irregularly shaped source (top)

Extended lenses



- 3D mass density $\rho(\vec{r})$ of an **extended lens** can be projected along the line of sight onto the lens plane to obtain the 2D **surface mass density** distribution: $\Sigma(\vec{\xi}) = \int_0^{D_s} \rho(\vec{r})dz,$

  where $\vec{r}$ is a 3D vector in space, and $\vec{\xi}$ is a 2D vector in the lens plane
- **Critical surface mass density** is given by the lens mass M "smeared out" over the area of the Einstein ring:

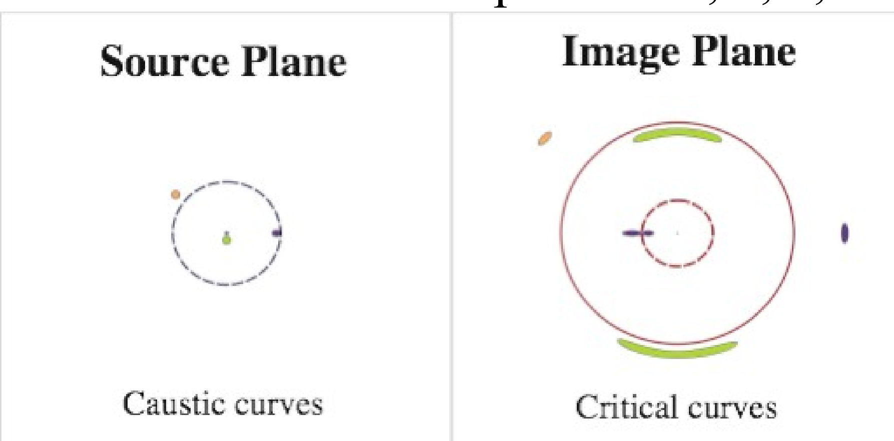
$$\Sigma_{cr} = \frac{M}{\pi \xi_E^2} = \frac{c^2}{4\pi G}\frac{D_s}{D_d D_{ds}} = 0.35\,\mathrm{g\,cm}^{-2}\left(\frac{D}{1\,\mathrm{Gpc}}\right)^{-1}$$

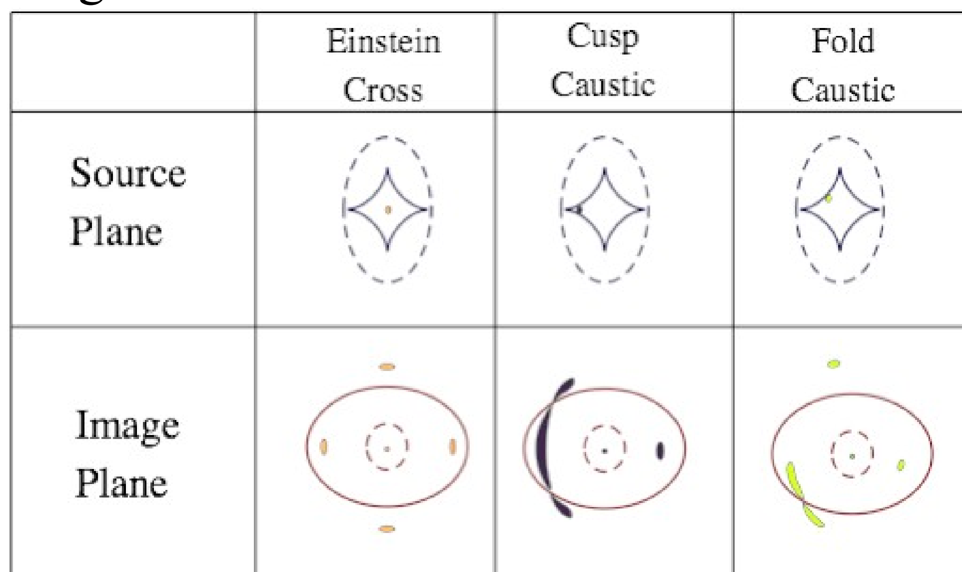
- **Dimensionless surface mass density (convergence) *κ*:** $\kappa(\vec{\theta}) := \frac{\Sigma(D_d\vec{\theta})}{\Sigma_{cr}}$
- A mass distribution for which $\kappa \geq 1$, i.e. $\Sigma \geq \Sigma_{cr}$ produces multiple images for some source positions (limit between "weak" from "strong" lenses)
- **Deflection potential *ψ*:** $\psi(\vec{\theta}) = \frac{1}{\pi}\int_{\mathbb{R}^2} \kappa(\vec{\theta}')\ln|\vec{\theta} - \vec{\theta}'|d^2\theta'$
- Deflection angle as a gradient of *ψ*: $\vec{\alpha} = \vec{\nabla}\psi(\vec{\theta}) \Rightarrow$ lens equation: $\boxed{\vec{\beta} = \vec{\theta} - \vec{\nabla}\psi(\vec{\theta})}$
- **Poisson equation:** $\nabla^2\Phi = 4\pi G\rho \Rightarrow \nabla^2\psi = 2\kappa$
- Different surface mass distributions ⇒ different deflection potentials ⇒ different extended lens models

Simple lens models

Lens Model	$\psi(\theta)$	$\alpha(\theta)$
Point mass	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4GM}{D_{\rm d}c^2} \ln\|\theta\|$	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4GM}{c^2 D_{\rm d}\|\theta\|}$
Singular isothermal sphere	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} \|\theta\|$	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2}$
Softened isothermal sphere	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} \left(\theta_{\rm c}^2 + \theta^2\right)^{1/2}$	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} \frac{\theta}{\left(\theta_{\rm c}^2 + \theta^2\right)^{1/2}}$

- Isothermal ellipsoid: $\psi(\theta_1, \theta_2) = \frac{D_{\rm ds}}{D_{\rm s}} 4\pi \frac{\sigma_v^2}{c^2} \left[\theta_{\rm c}^2 + (1-\epsilon)\theta_1^2 + (1+\epsilon)\theta_2^2\right]^{1/2}$, where ε measures the ellipticity
- Extended lenses can produce 3, 4, 5, … images of a source



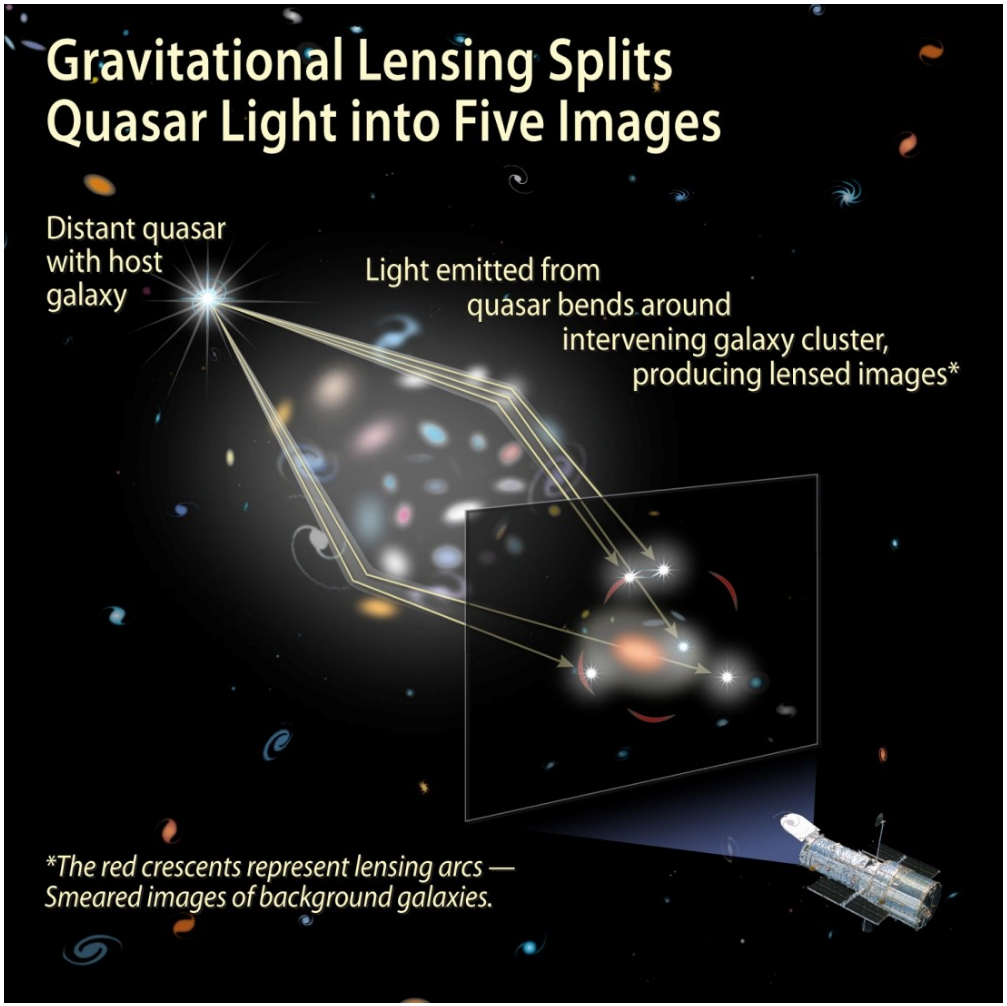


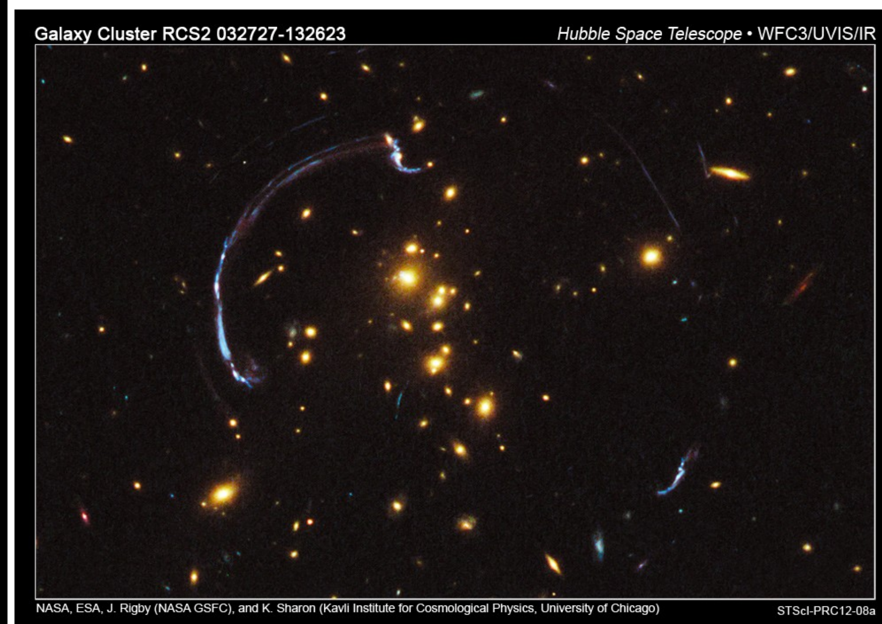
Images due to SIS (top) and non-singular isothermal ellipsoid (right) lens models

Gravitational lensing applications

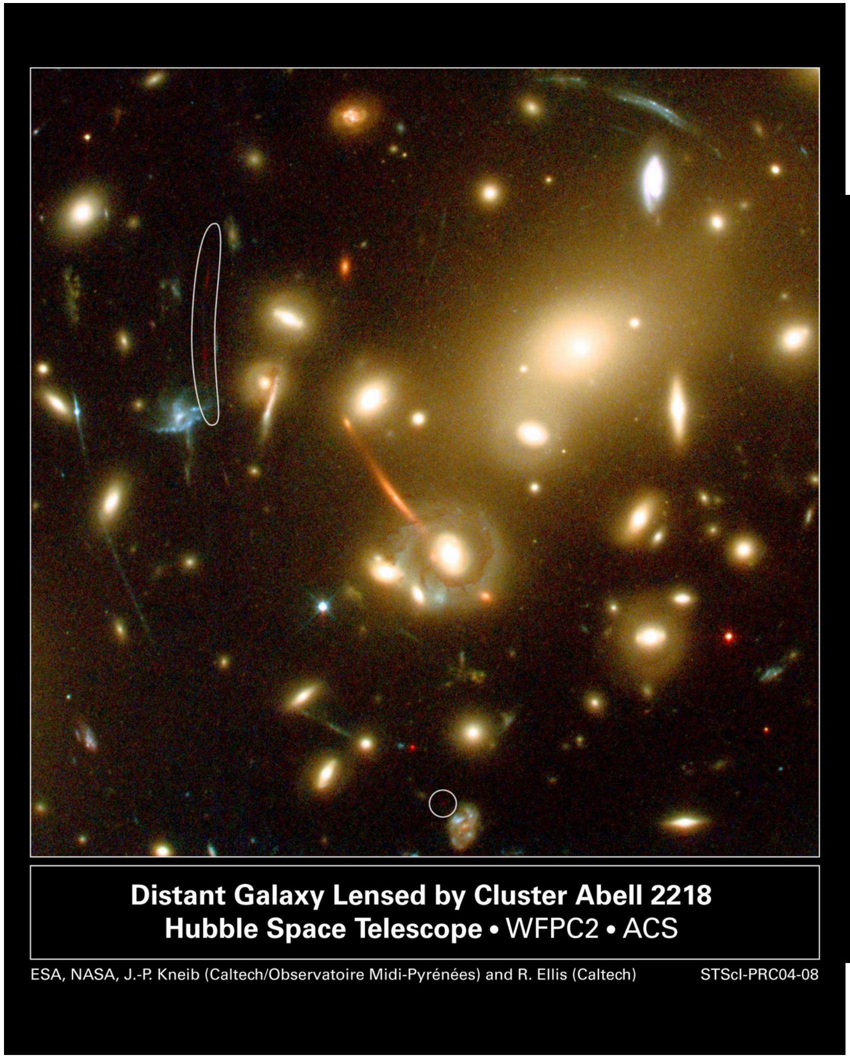
1. **Strong lensing:**
   - by galaxies (**macrolensing**) - multiple images of the background sources: determination of cosmological parameters (H_0 from time delays, Ω_M, Ω_Λ, Ω_k from lensing statistics)
   - by stars (**microlensing**) - amplification (magnification) of the background sources: detection of extrasolar planets, studying the innermost regions of active galaxies around their central supermassive black holes, constraining cosmological parameters
   - by clusters of galaxies - giant arcs as images of distant background galaxies: finding the most distant galaxies in the Universe (natural telescopes)
2. **Weak lensing:**
   - by foreground matter with lower density distribution - shape distortions of the background sources: the only direct mean to detect the dark matter, studying the distribution of visible and dark matter in the Universe

Strong lensing by galaxy clusters (natural telescopes): finding the most distant galaxies

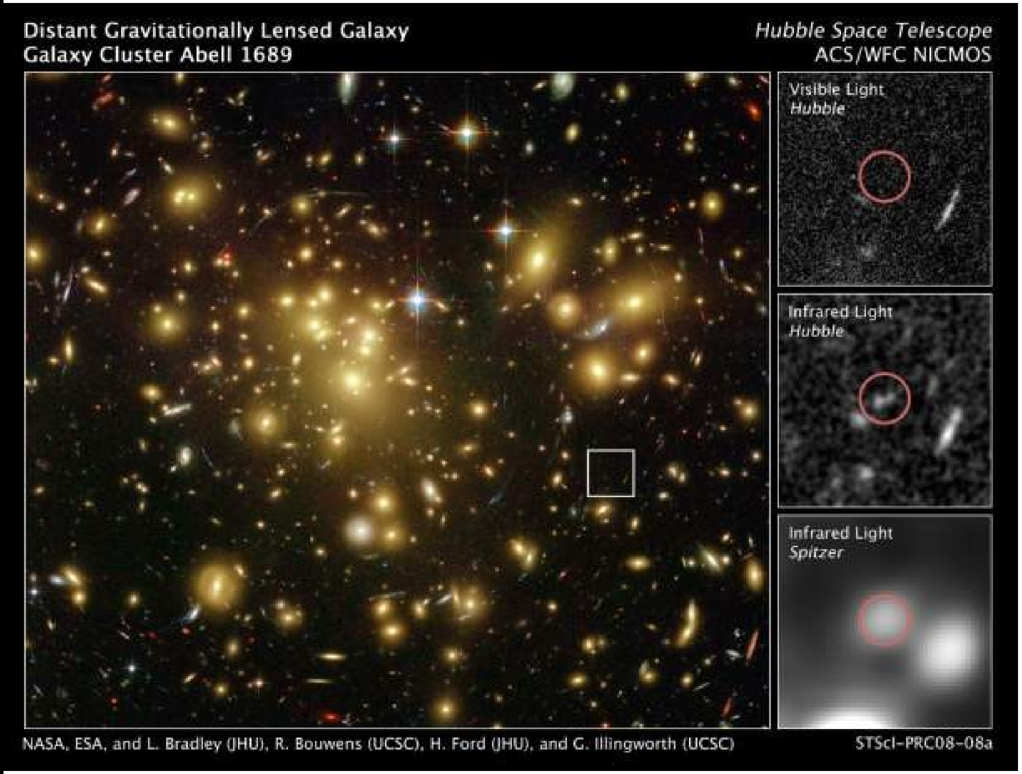




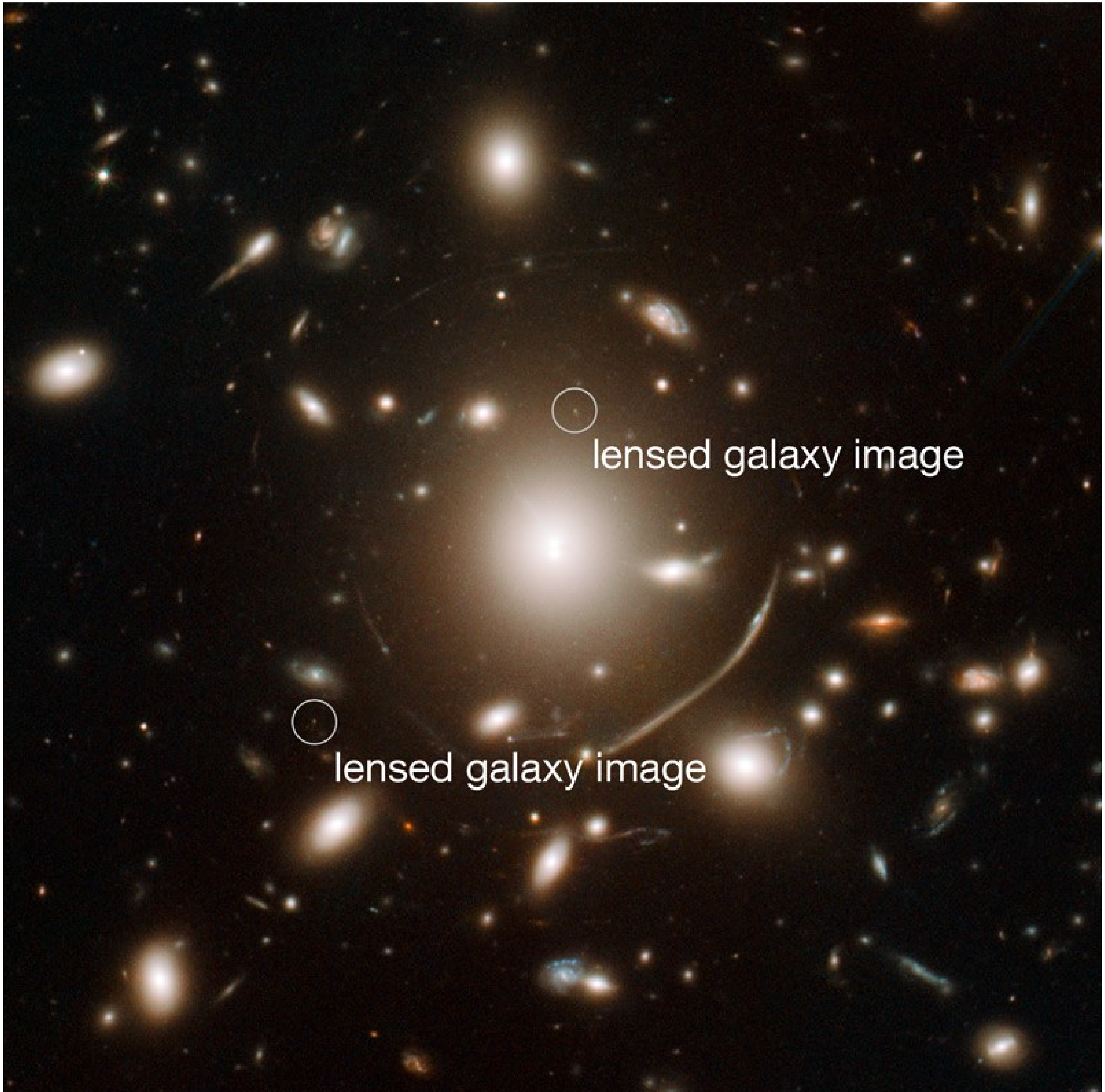
Finding the most distant galaxies



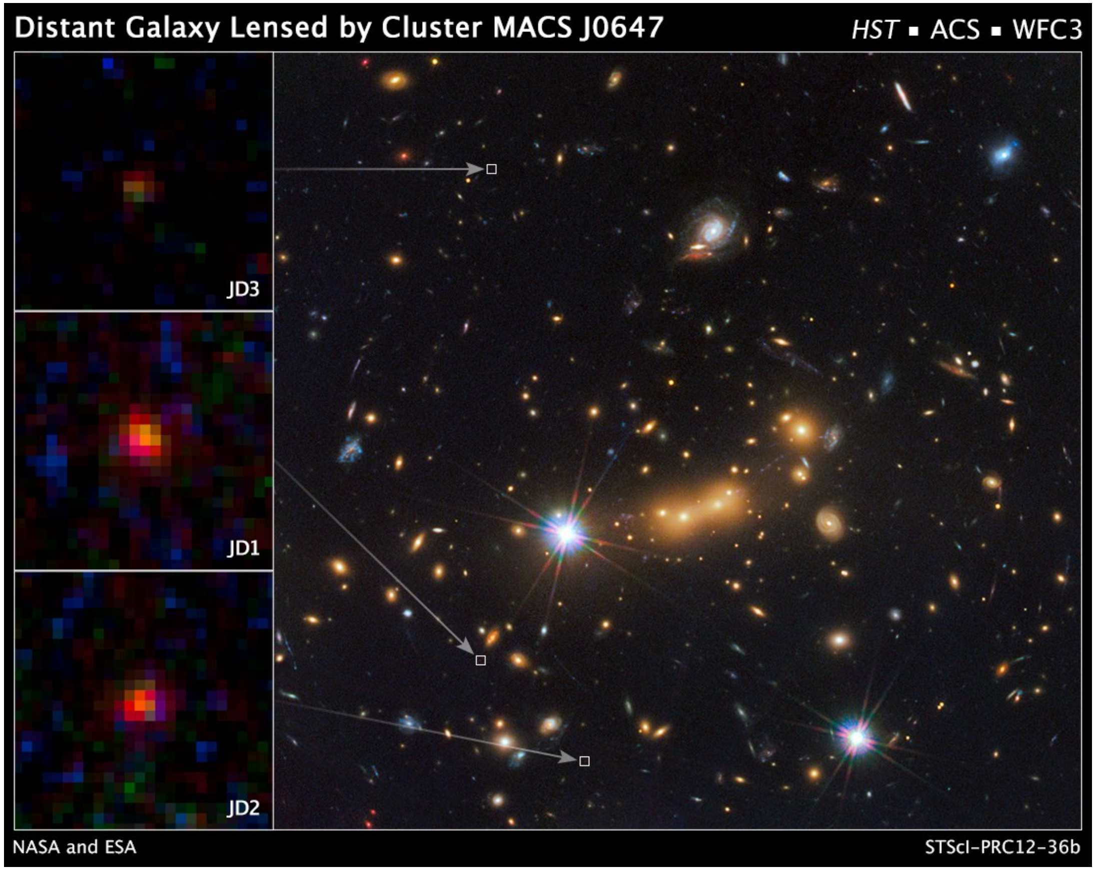
Red arc and point: the most distant galaxy known until 2004, located at $z \sim 7$ ($\approx 13 \times 10^9$ ly)



The most distant galaxy known until 2008, located at $z \sim 7.6$ ($\approx 13 \times 10^9$ ly)

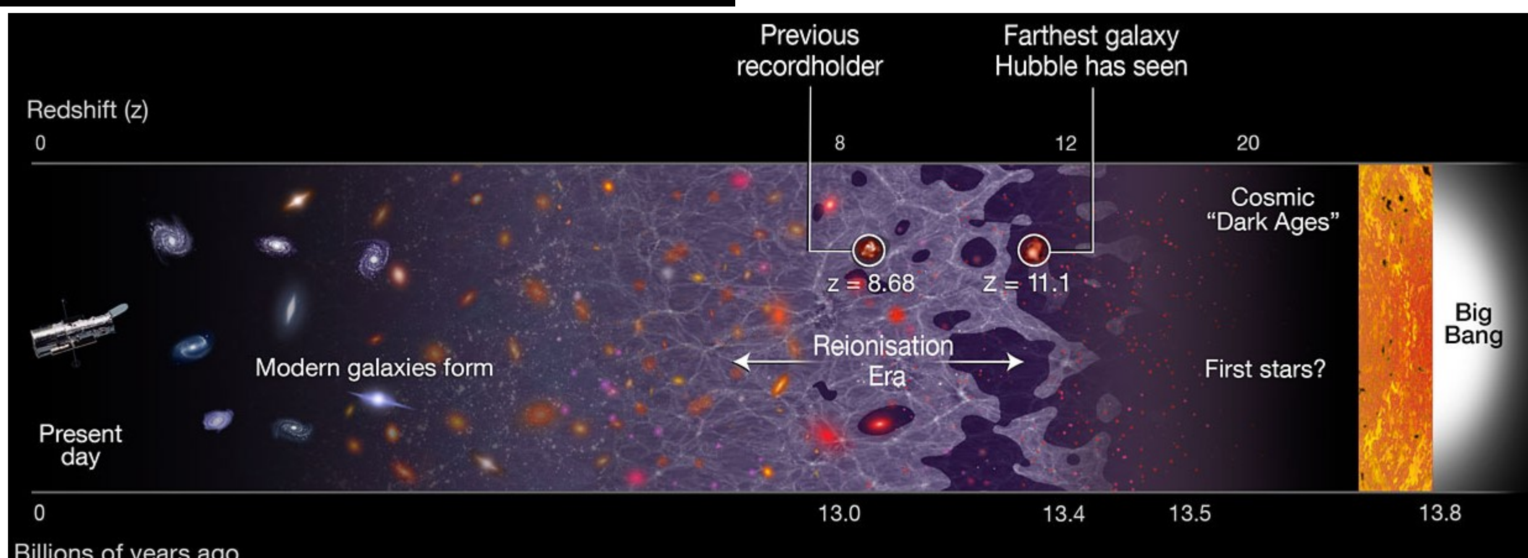


The most distant galaxy known until 2011, formed 13.5 billion years ago, discovered by lensing effect by galactic cluster Abell 383

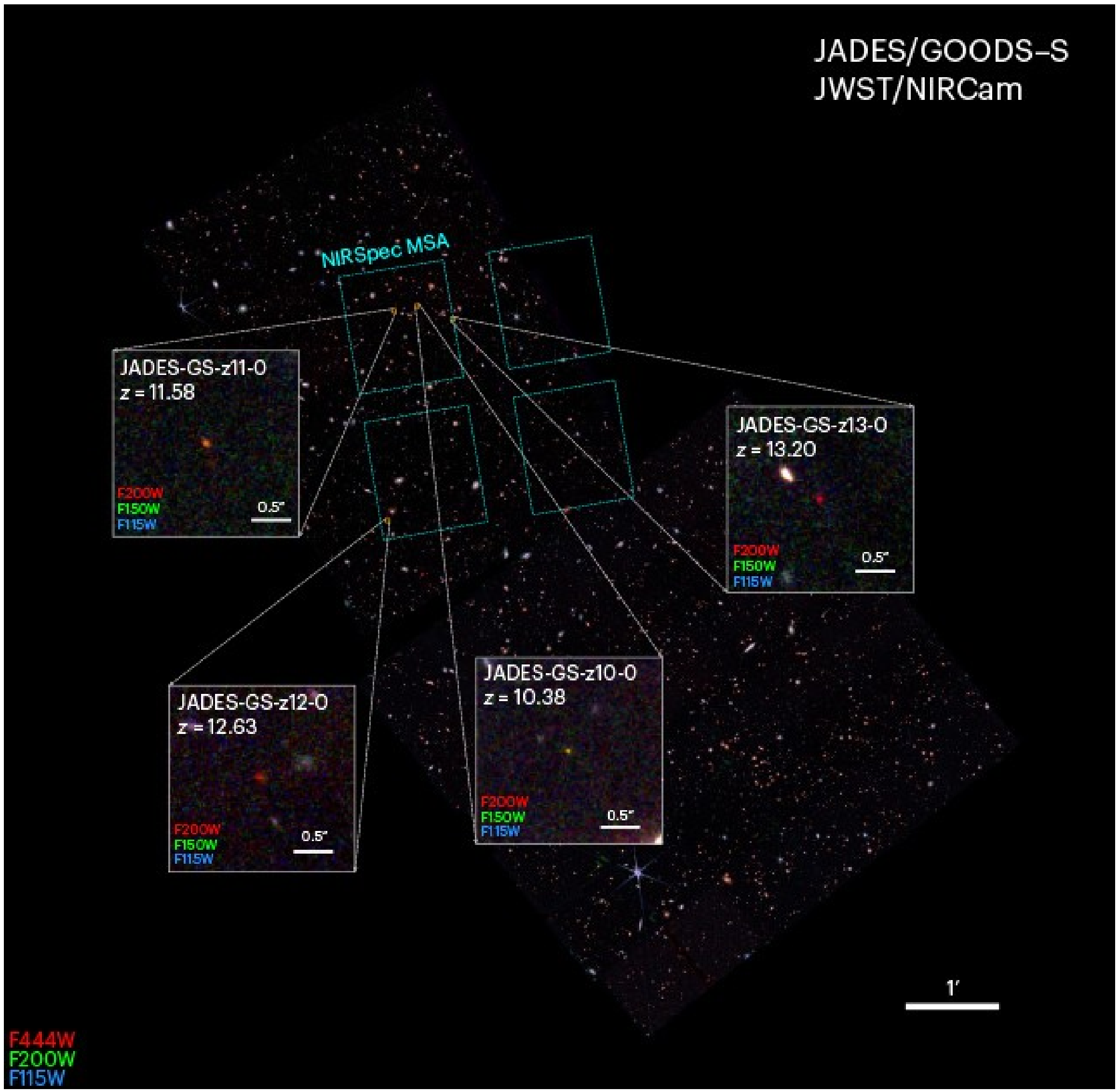


MACS0647-JD: the most distant galaxy discovered in 2012: the distance to the cluster is 5.6 billion ly ($z = 0.591$) and to the lensed galaxy is 13.3 billion ly ($z = 11$)

- Current record from 2016: galaxy GN-z11 at $z = 11.1$



Four most distant galaxies ever seen detected by JWST in 2023



Gravitational lensing by galaxy cluster Abell 2744 (Pandora's Cluster)

Robertson et al. 2023, Nature Astronomy, 7, 611.

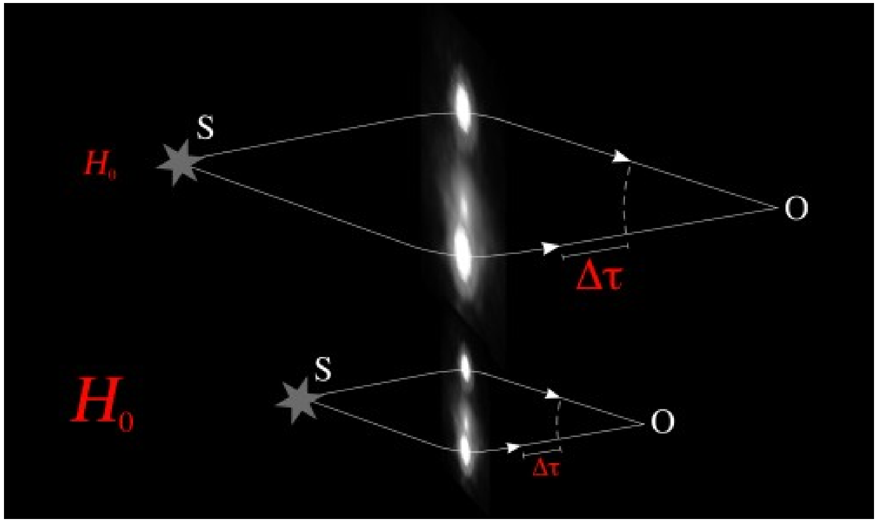
Fermat potential and lensing time delay

- **Fermat potential**: $\tau(\vec{\theta},\vec{\beta}) = \frac{1}{2}(\vec{\theta}-\vec{\beta})^2 - \psi(\vec{\theta})$ is, up to an affine transformation, the **light travel time** along a ray starting at position $\vec{\beta}$, traversing the lens plane at position $\vec{\theta}$ and arriving at the observer
- **Fermat principle:** the physical light rays are those for which the light travel time is stationary, and lens equation is then: $\vec{\nabla}\tau(\vec{\theta},\vec{\beta}) = 0$
- **Light travel time (**physical time delay function**)** for a lensed image:

$$\tau(\vec{\theta},\vec{\beta}) = \tau_{\rm geom} + \tau_{\rm grav} = \frac{D_{\Delta t}}{c}\left(\frac{1}{2}(\vec{\theta}-\vec{\beta})^2 - \psi(\vec{\theta})\right), \quad D_{\Delta t} = (1+z_d)\frac{D_d D_s}{D_{ds}},$$

where $\tau_{\rm geom}$ is extra path length between observer and source, while $\tau_{\rm grav}$ is retardation due to gravitational potential (Shapiro delay)

- $D_{\Delta t}$ is the **time-delay distance** which is $\propto H_0^{-1}$ and very weakly sensitive to Ω_M and Ω_Λ
- $\Delta t = \tau_2 - \tau_1 \Rightarrow D_{\Delta t} \Rightarrow H_0$
- Estimates of H_0 depend on lens model (i.e. on deflection potential ψ)



Measuring the time delays from the light curves of the images

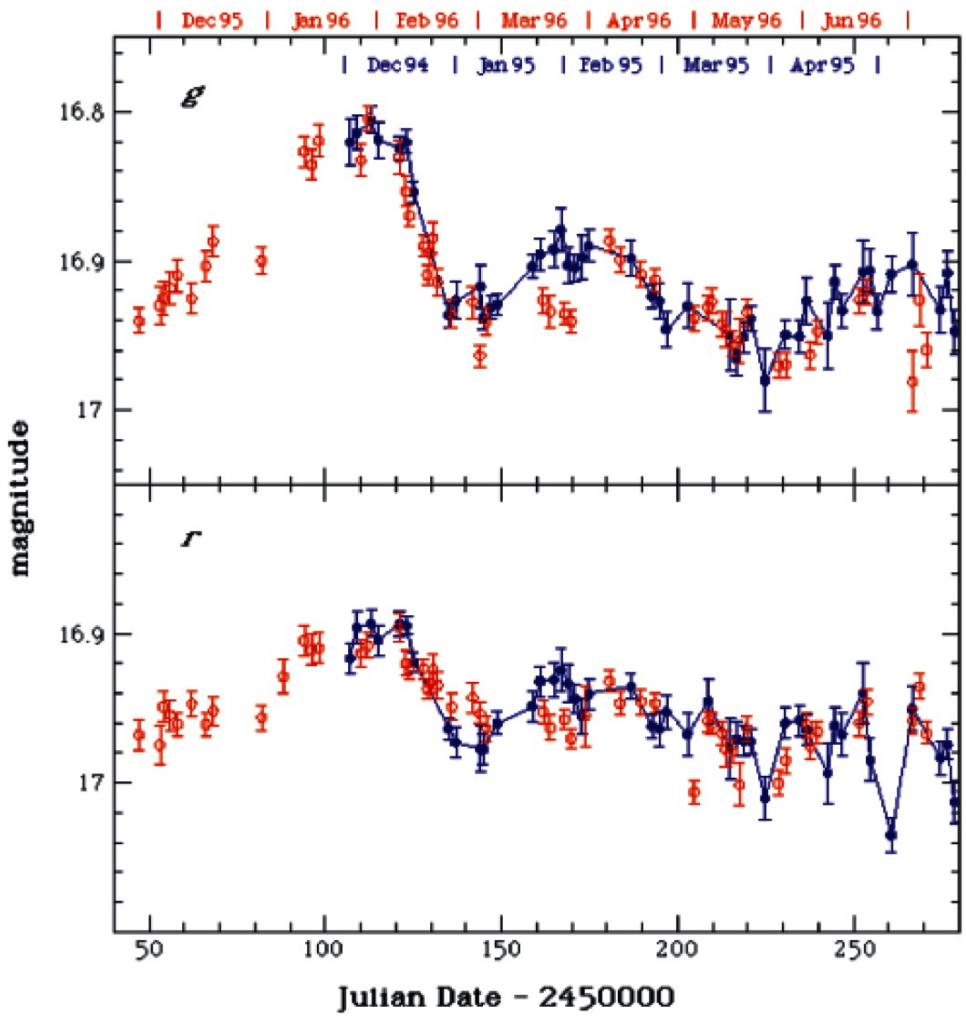
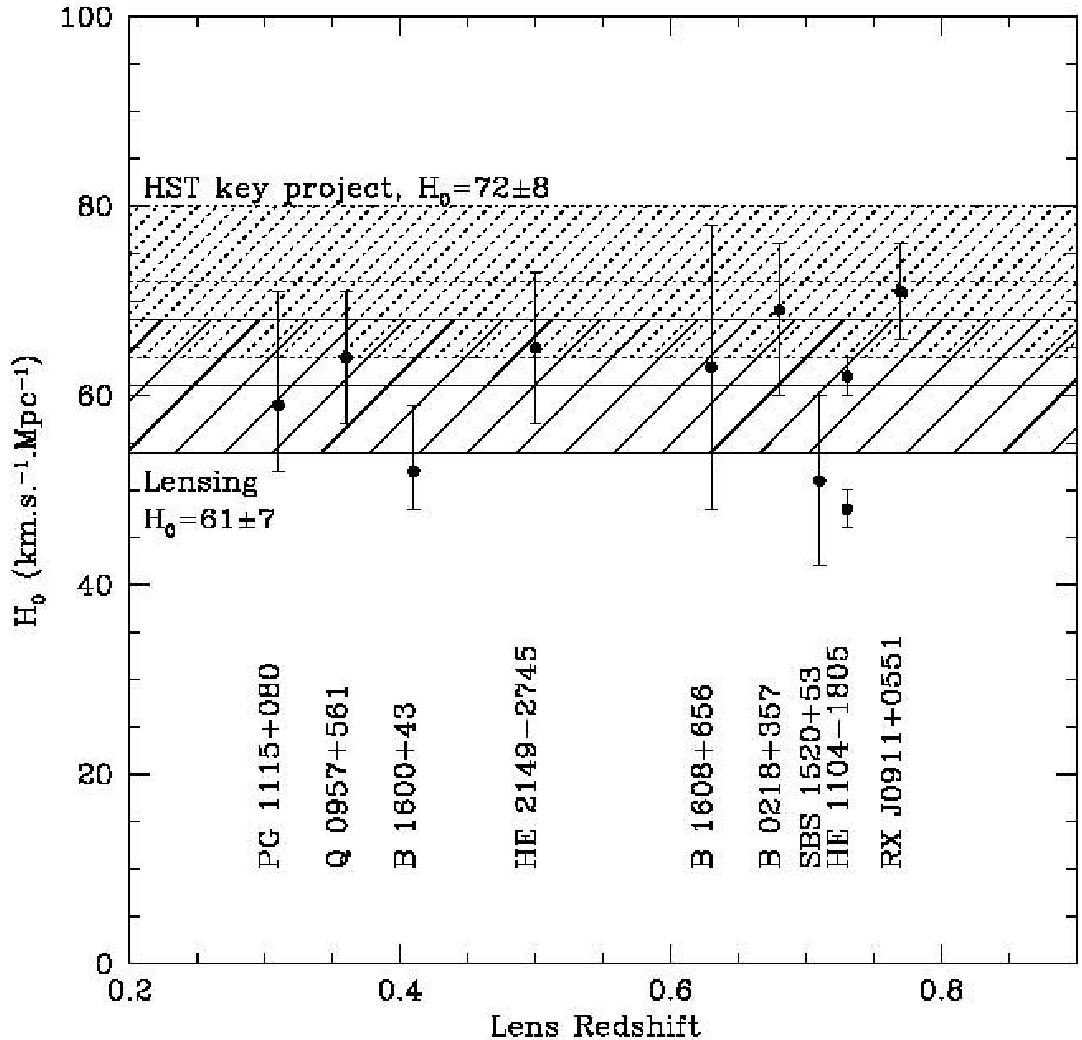


Fig. 9. Light curves of the two images of the QSO 0957+561A,B in two different filters. The two light curves have been shifted in time relative to each other by the measured time delay of 417 days, and in flux according to the flux ratio. The sharp drop measured in image A in Dec. 1994 and subsequently in image B in Feb. 1996 provides an accurate measurement of the time delay (data from Kundić et al. 1997)

Determining H_0 from lensing time delays

Table 1.1. *Time Delay Measurements*

System	N_{im}	Δt (days)
HE1104–1805	2	161 ± 7
PG1115+080	4	25 ± 2
SBS1520+530	2	130 ± 3
B1600+434	2	51 ± 2
HE2149–2745	2	103 ± 12
RXJ0911+0551	4	146 ± 4
Q0957+561	2	417 ± 3
B1608+656	4	77 ± 2
B0218+357	2	10.5 ± 0.2
PKS1830–211	2	26 ± 4
B1422+231	4	(8 ± 3)

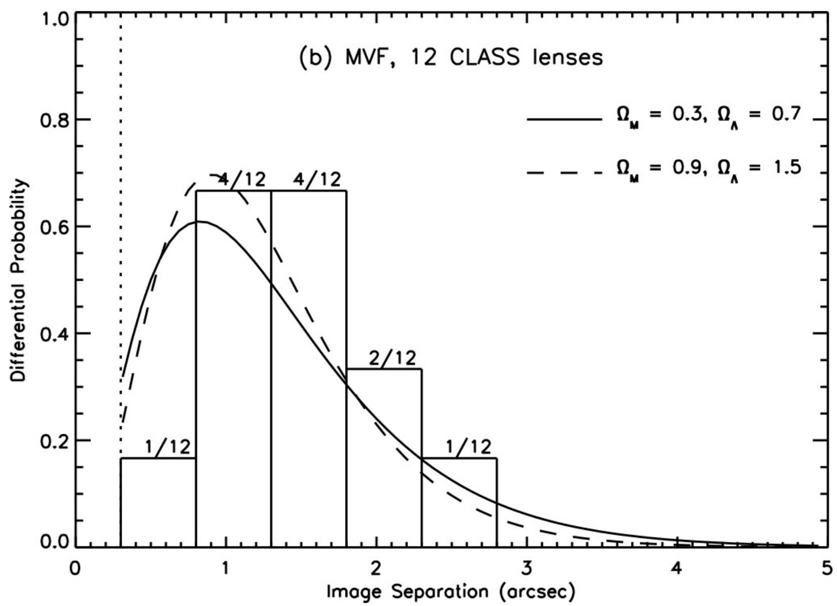


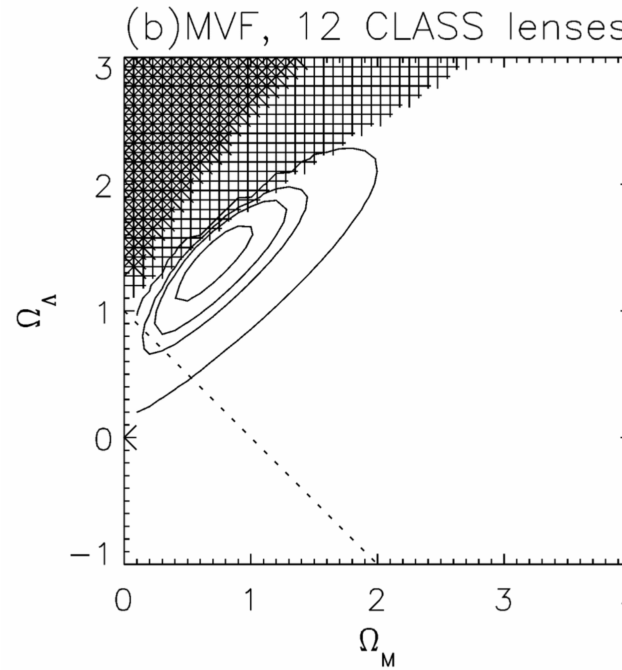
Schechter, 2005, IAU Symposium 225, 281

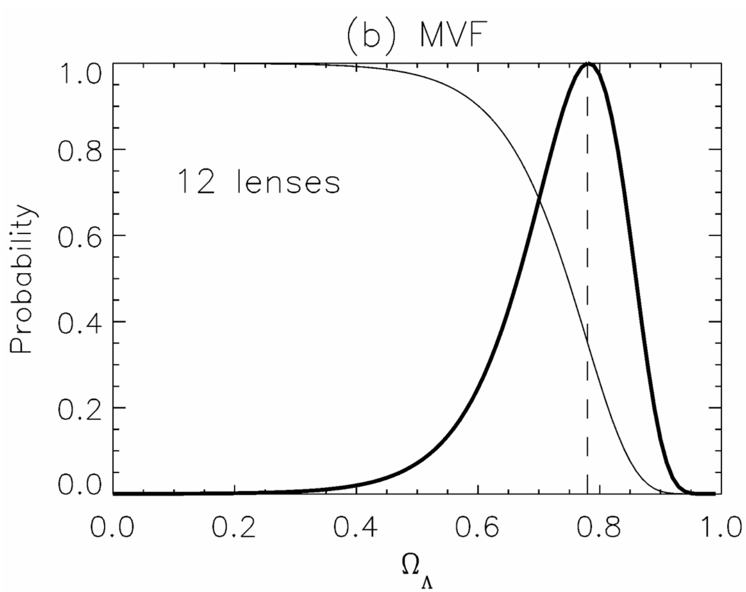
Optical depth and statistics of strong lenses

- **Optical depth (τ)** - the probability of a source being lensed
- τ is the chance of seeing a lensing event, i.e. the probability that at any instant of time a source is within the Einstein ring of a lens
- **Cross section of strong lensing (A)** - area in the lens plane where the separation between the lens and source is sufficiently small for strong lensing to occur (fraction of the sky in which you can place a source and see the lensing effect): $A = \pi\theta_E^2$
- The total τ is obtained by summing the cross sections of all deflectors between the observer and source, and it depends on cosmological parameters through θ_E
- **Statistical distributions** obtained from differential optical depth $d\tau$: distribution per image separations $\Delta\theta$ $(d\tau/d\Delta\theta)$, distribution per redshift of lens galaxies z_l $(d\tau/dz_l)$ and joint distribution $d^2\tau/(dz_l\, d\Delta\theta)$ per both z_l and $\Delta\theta$
- **Relative probability** of finding a lens at some z_l: $\delta p_l = \frac{d\tau}{dz_l}/\tau$
- Fitting a probability distribution from an observed sample of strong lenses by a modeled prediction $\Rightarrow \Omega_M \wedge \Omega_\Lambda$ (the results do not depend on H_0)
- It is essentially a comoving volume cosmological test

Statistics of strong lenses: early results







- Left: relative differential probability (thick curve) and cumulative probability (thin curve) for lensing in spatially flat cosmology (Mitchell et al. 2005, *ApJ*, 622, 81)
- Early results were not in agreement with other cosmological tests

Impact of Gravitational Lensing on Cosmology
Proceedings IAU Symposium No. 225, 2004
Mellier, Y. & Meylan, G. eds.

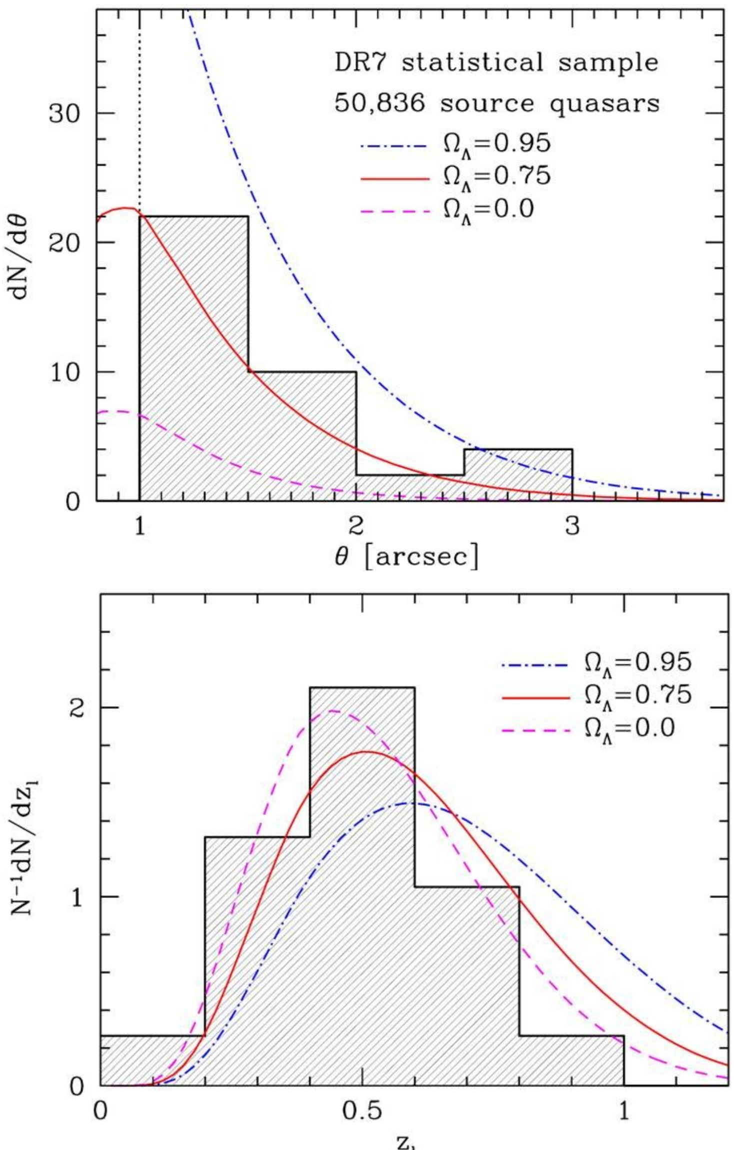
doi:10.1017/S1743921305002231

Quasar Lensing Statistics and Ω_Λ: What Went Wrong?

Dan Maoz

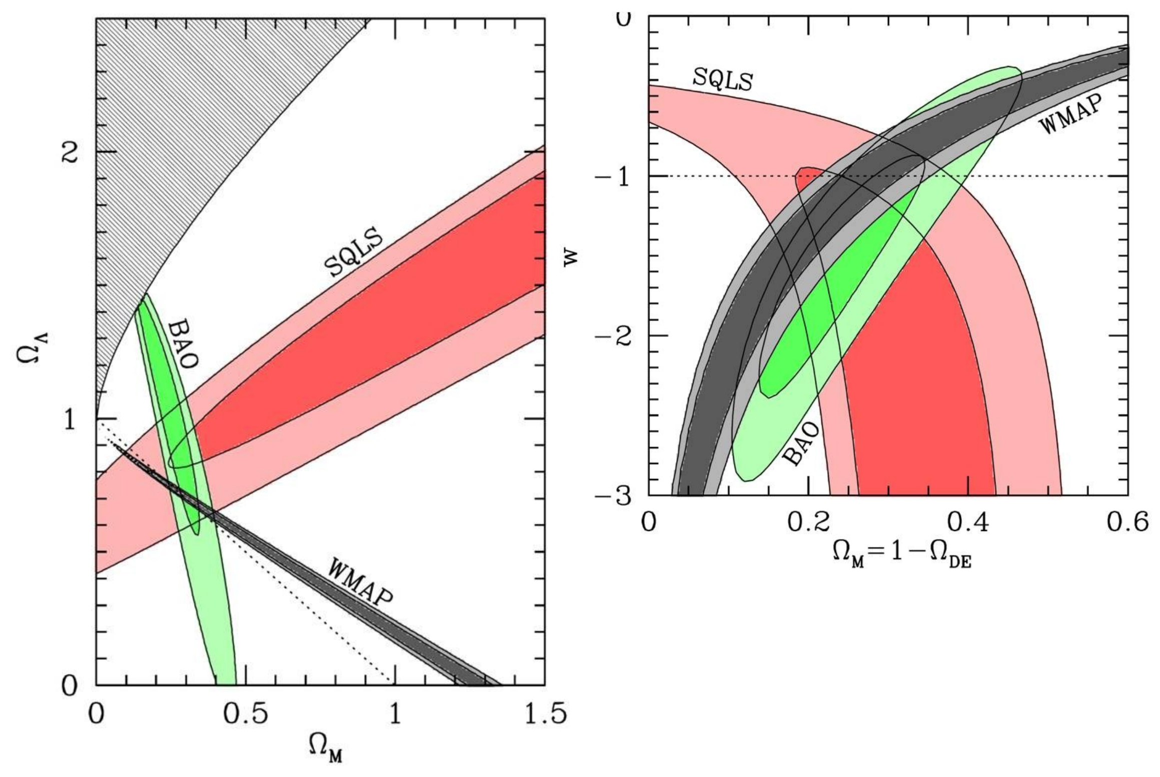
Statistics of strong lenses: more recent results

- More recent results (Cao et al. 2012, *ApJ*, 755, 31; Oguri et al. 2012, *AJ*, 143 120) are in agreement with other cosmological tests

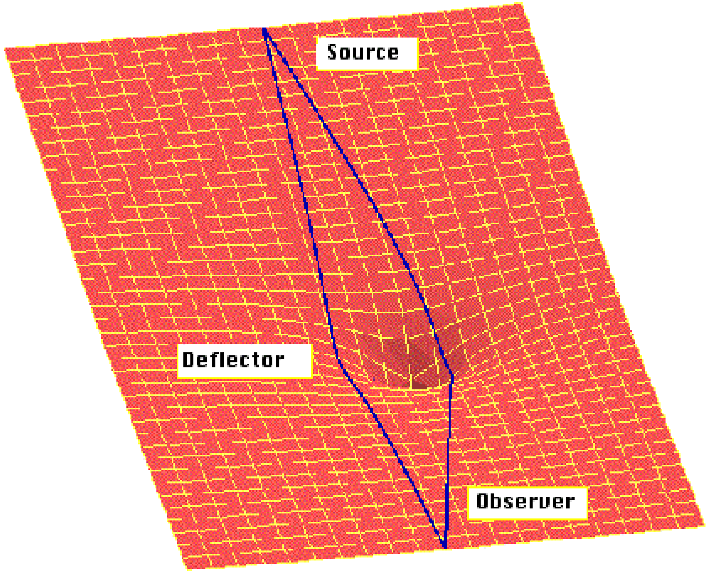


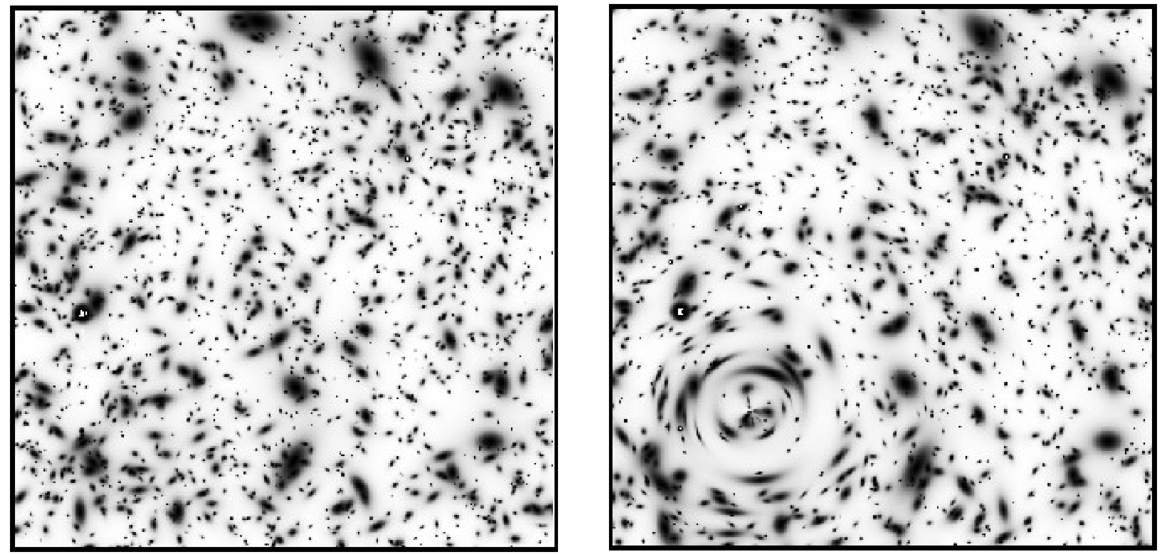
Left: image separation distribution (top) and normalized lens redshift distribution (bottom) of the strong lenses in the statistical lens sample

Bottom: 1σ and 2σ confidence regions for Ω_M , Ω_Λ and w

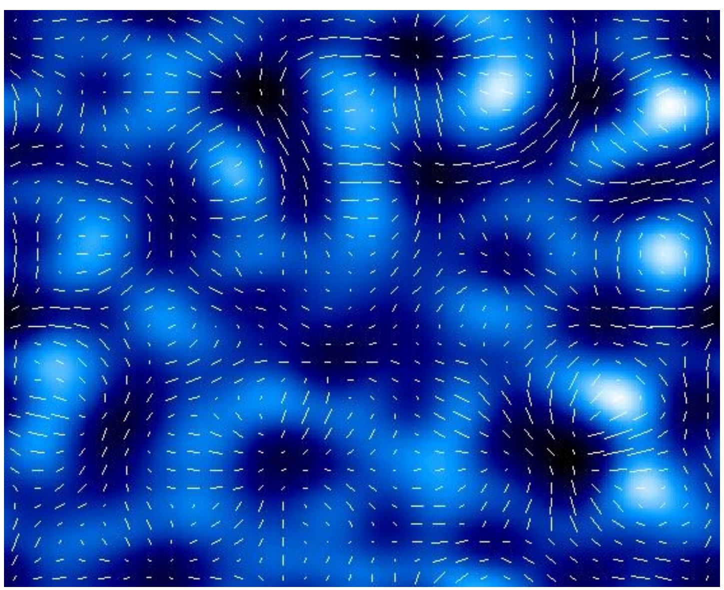


Weak lensing

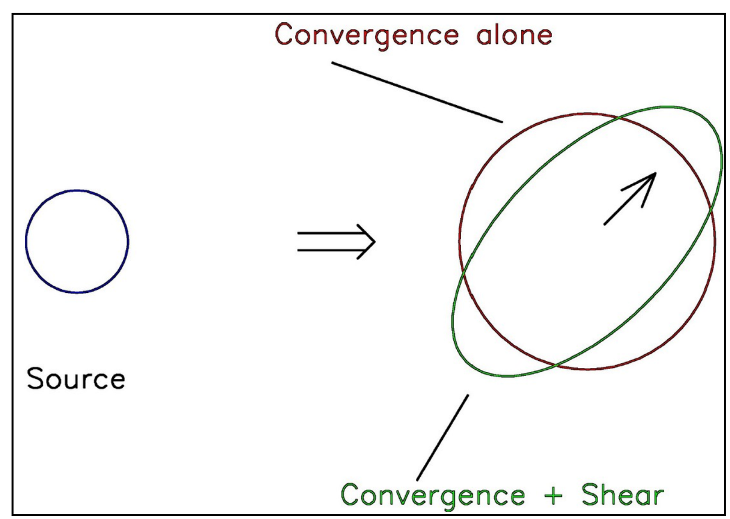


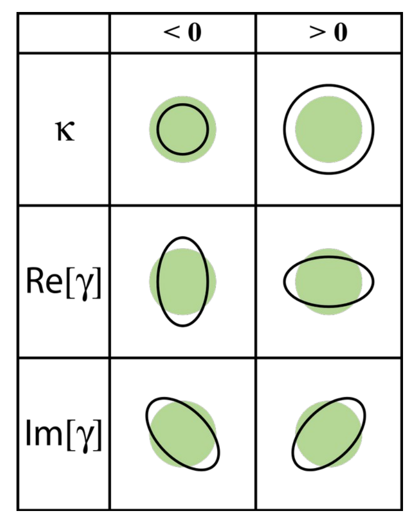


Background galaxies in absence of lensing (left), and in the case of weak lensing by invisible foreground cluster of galaxies (right)



Projected mass (shades of blue) and shear (white sticks)





Mass reconstructions by weak lensing

- An observed galaxy ellipticity is a combination of its intrinsic ellipticity and shear γ
- Shear γ can be estimated by averaging over many galaxy images, assuming that the intrinsic ellipticities are *randomly oriented*
- **Mass reconstruction:** obtaining the surface mass density κ from the measured values of shear γ

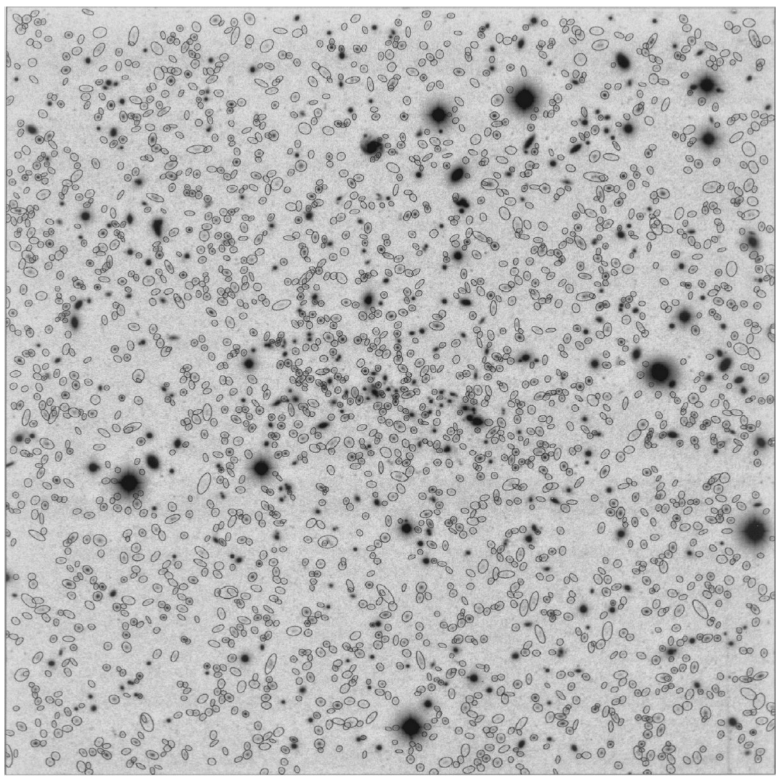


FIG. 3.—Full 2048 × 2048 pixel *I*-band CCD image of MS 1054−03 with the ellipses drawn around all the 2395 objects in the $I > 21.5$ catalog

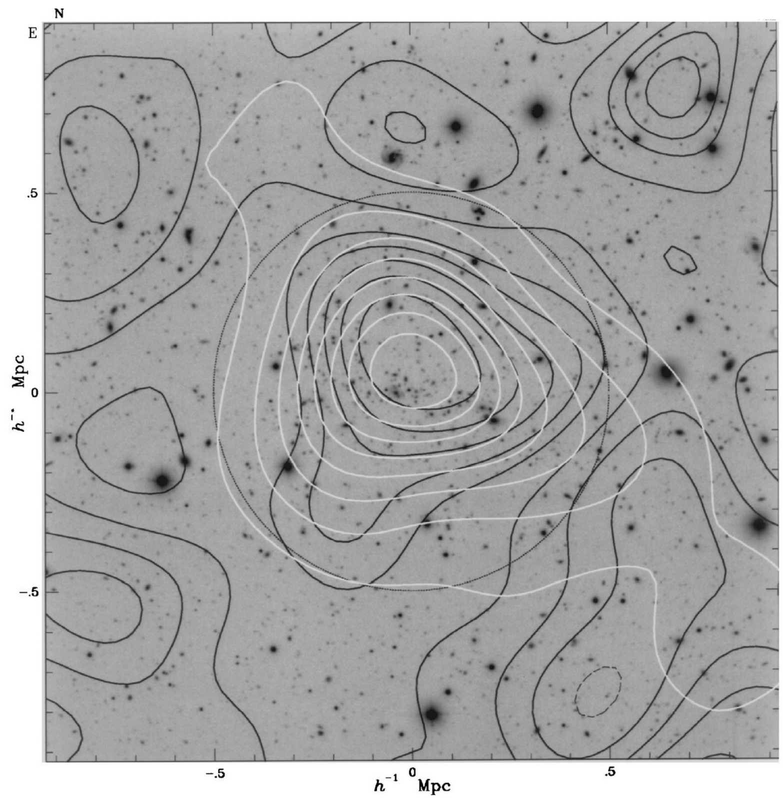
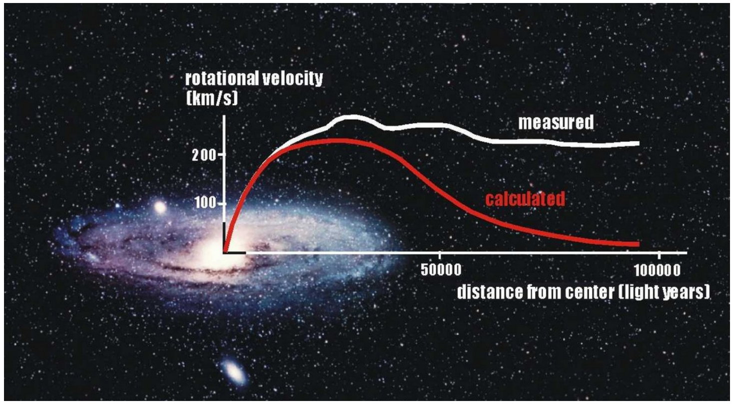


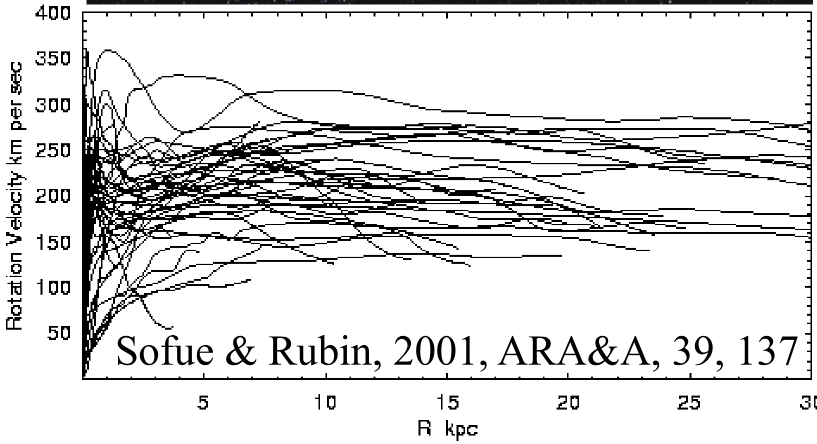
FIG. 5.—Contour plot of the surface mass density (*black contour lines*) and cluster light distribution (*white contour lines*) overlaid on the 2048^2 pixel optical image of the cluster.

Left: image of cluster of galaxies MS1054−03 with about 2400 measured galaxy ellipticities
Right: mass reconstruction (black), compared to light distribution (white)

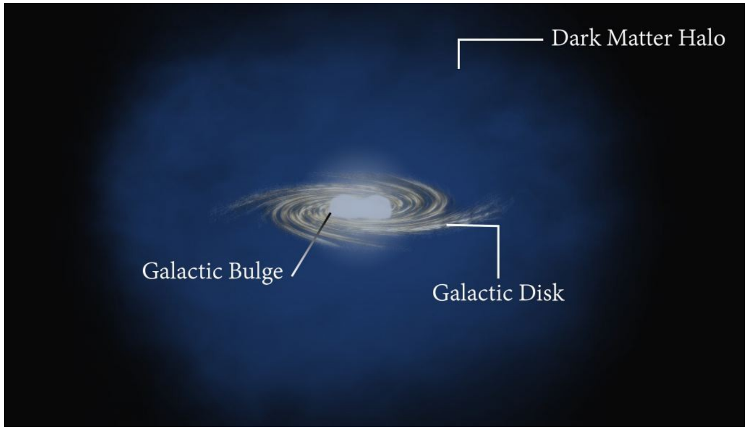
Dark matter

- Zwicky applied virial theorem to the motions of galaxies in the Coma Cluster ⇒ several hundred times more estimated than observable mass ⇒ "dunkle Materie" (Zwicky, 1933, HPA, 6, 110)
- Vera Rubin in the 1960s and 1970s: the observed rotation curves of spiral galaxies are flat ⇒ 6 times as much dark as visible mass
- DM is composed from non-baryonic particles which are so weakly interacting that they move purely under the influence of gravity ⇒ it can be directly detected only by weak lensing

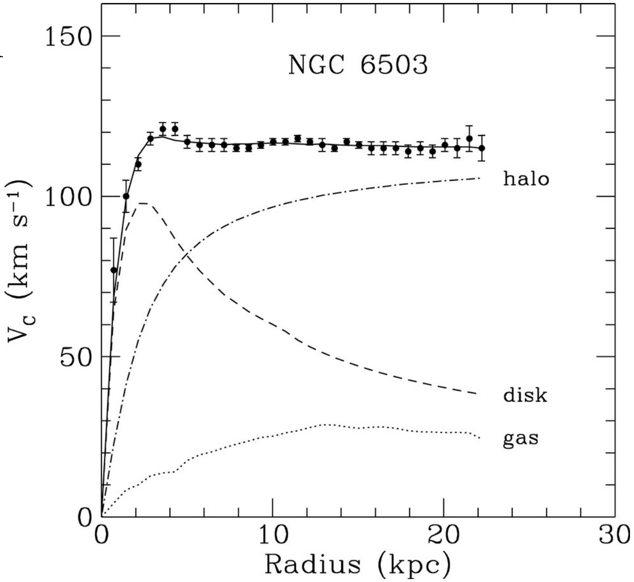




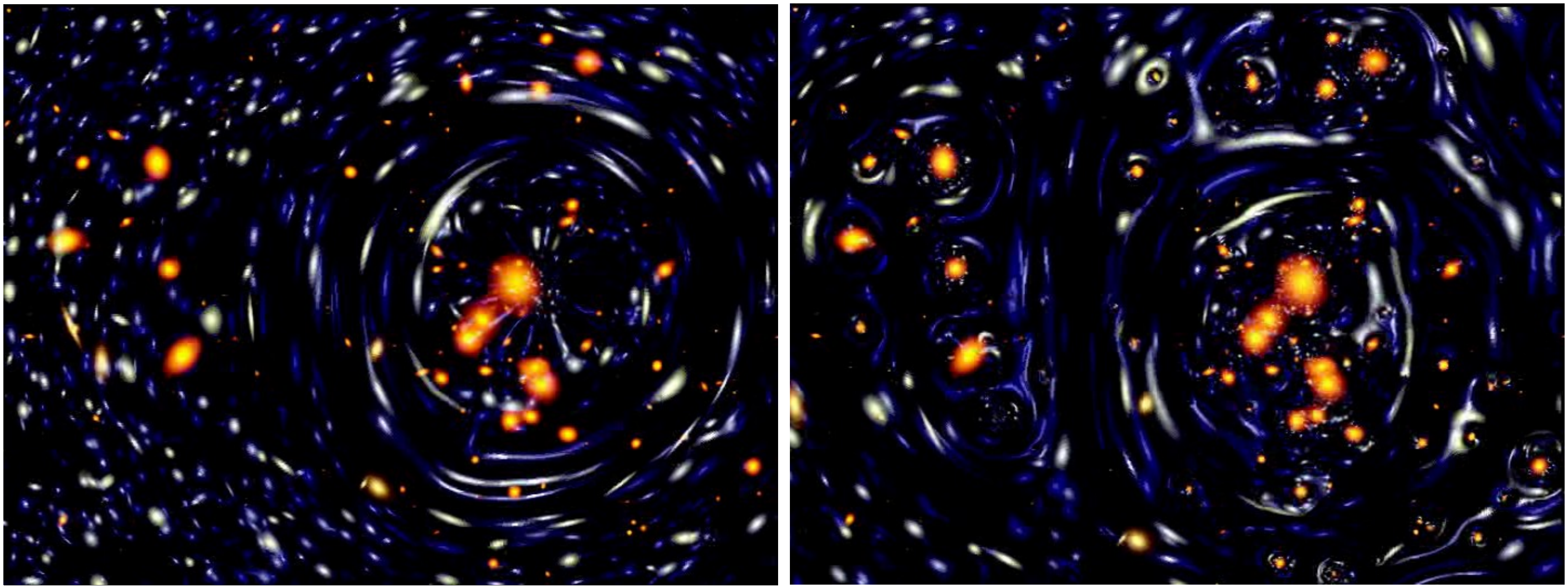
- Hypothesis: a spherical **dark matter halo** around a spiral galaxy (Navarro, Frenk & White, 1996, ApJ, 462, 563):



$$\rho(r) = \frac{\rho_0}{\frac{r}{r_0}\left(1 + \frac{r}{r_0}\right)^2}$$

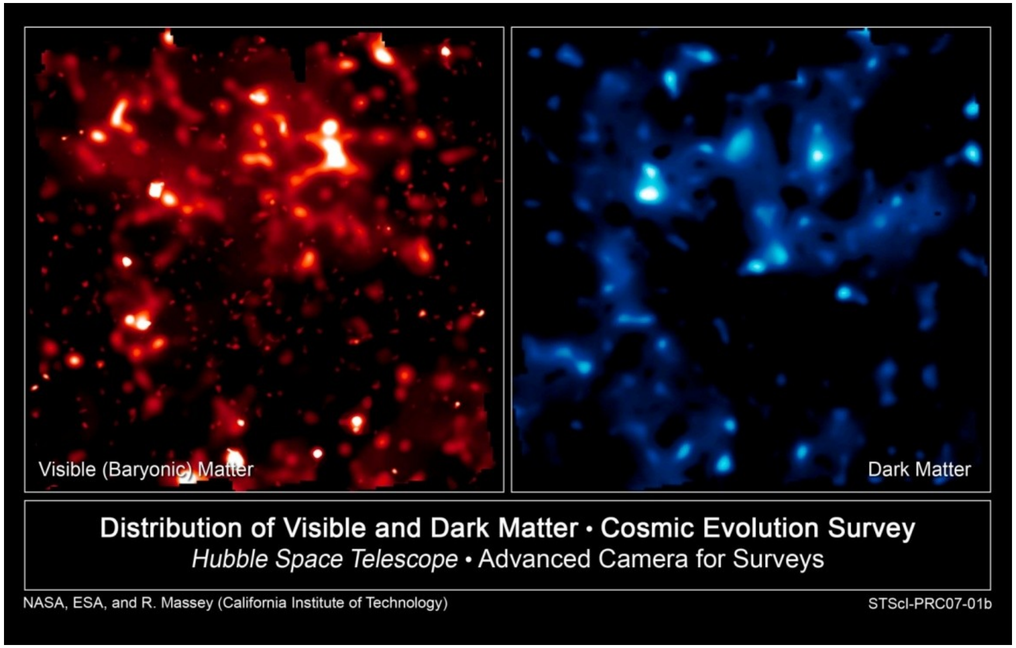


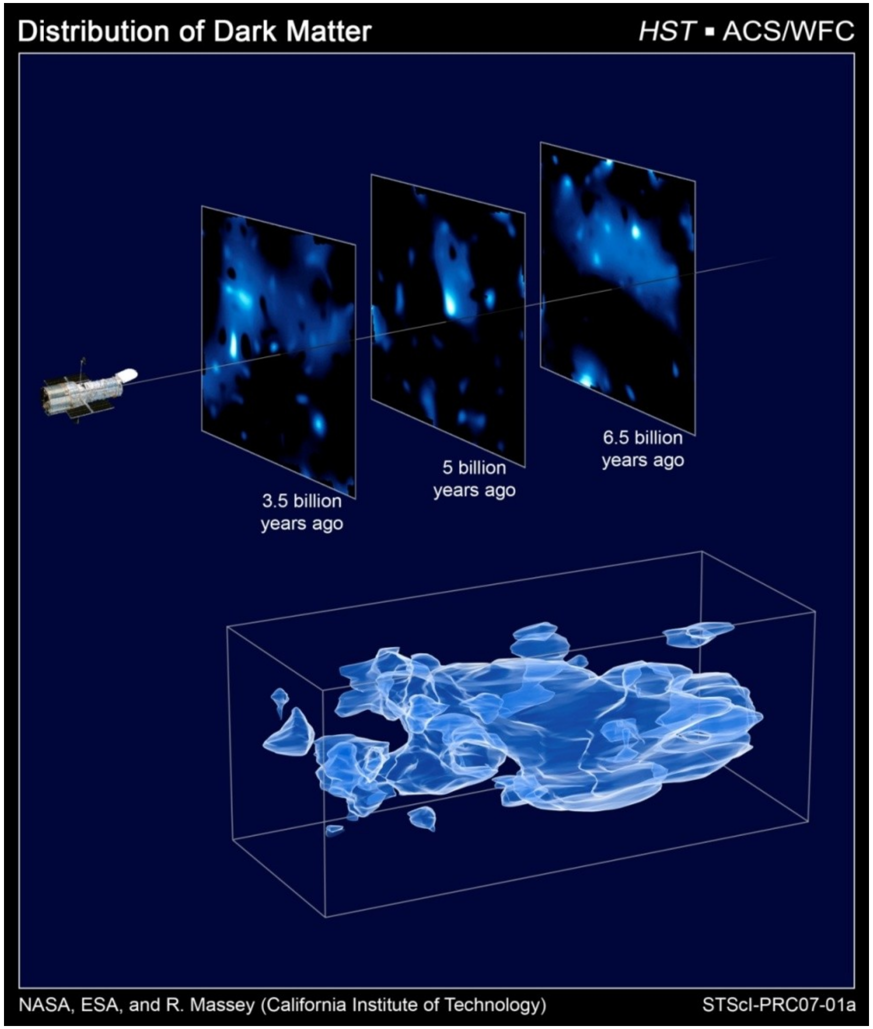
Gravitational lensing by dark matter



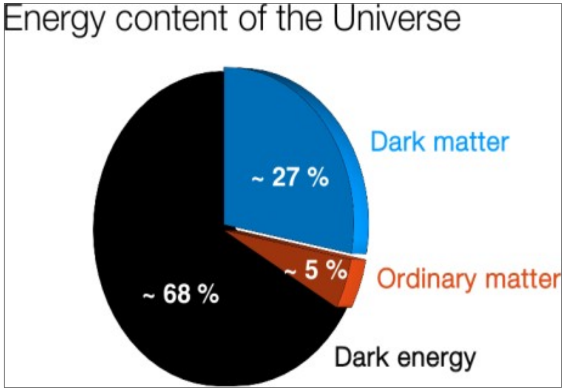
Gravitational lensing by a massive cluster of galaxies with two different distributions of the same amount of the dark matter over the cluster (orange), causing a particular distortion of the background galaxies (white and blue).

Spatial distribution of dark matter





- Dark matter constitutes about 5/6 of the total mass of the Universe



Detection of dark matter in the case of "Bullet Cluster" (1E 0657-558)

THE ASTROPHYSICAL JOURNAL, 648:L109–L113, 2006 September 10

A DIRECT EMPIRICAL PROOF OF THE EXISTENCE OF DARK MATTER[1]

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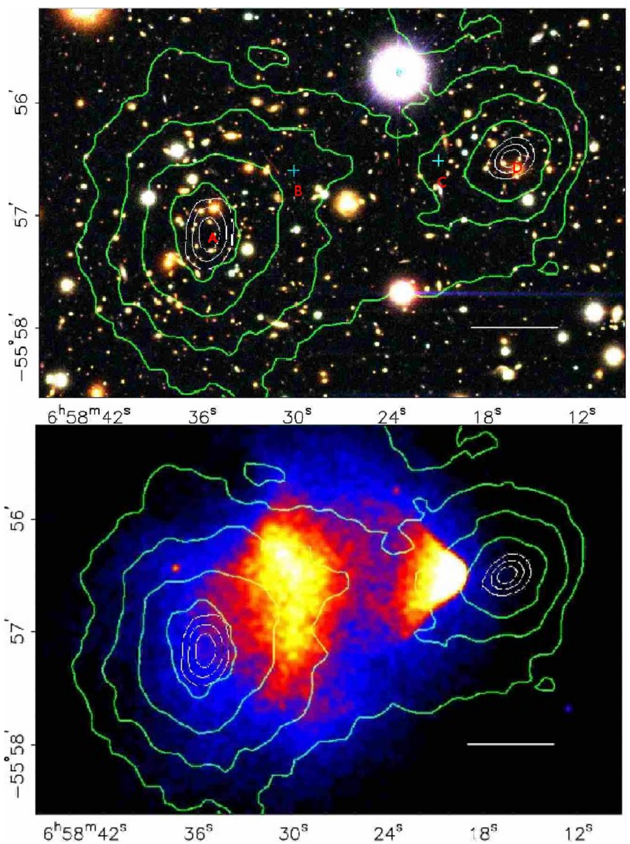
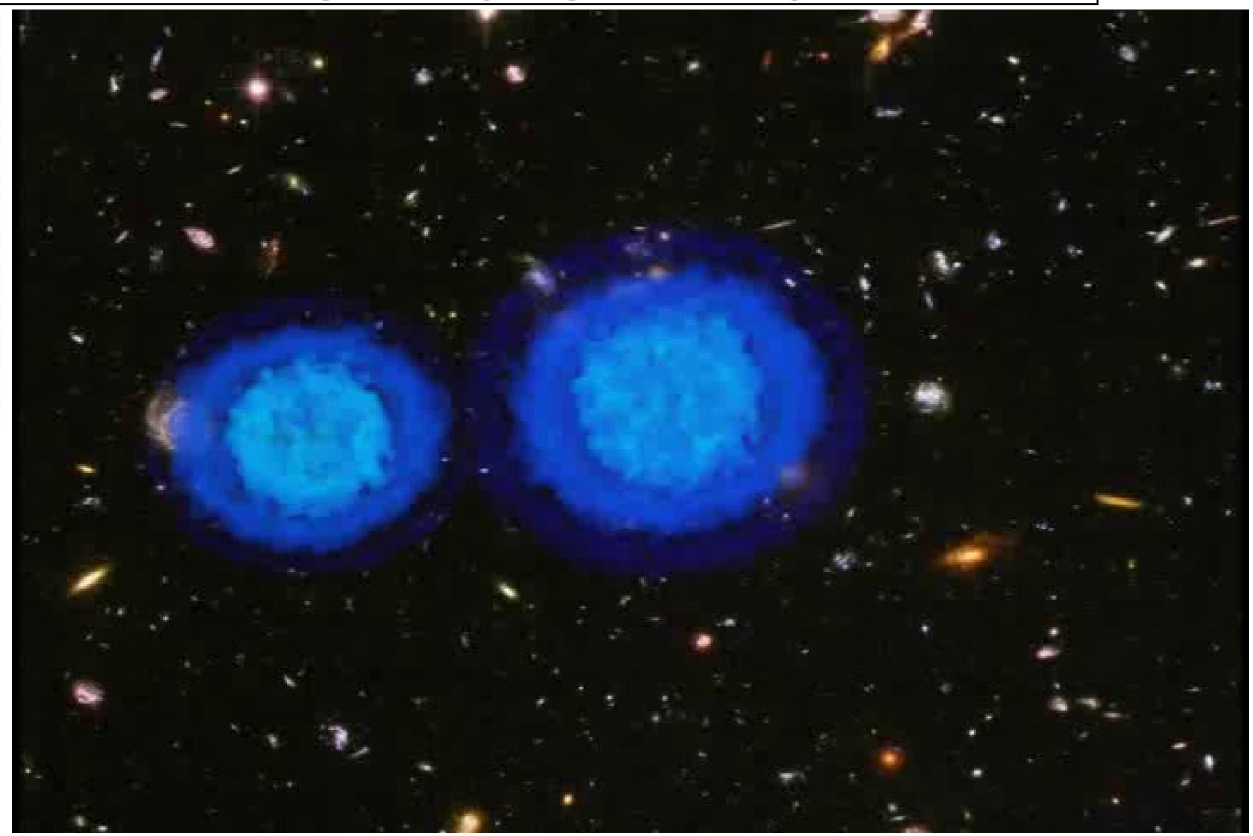
Received 2006 June 6; accepted 2006 August 3; published 2006 August 30





Counterexample: Abell 520

The Astrophysical Journal, 668:806–814, 2007 October 20

A DARK CORE IN ABELL 520[1]

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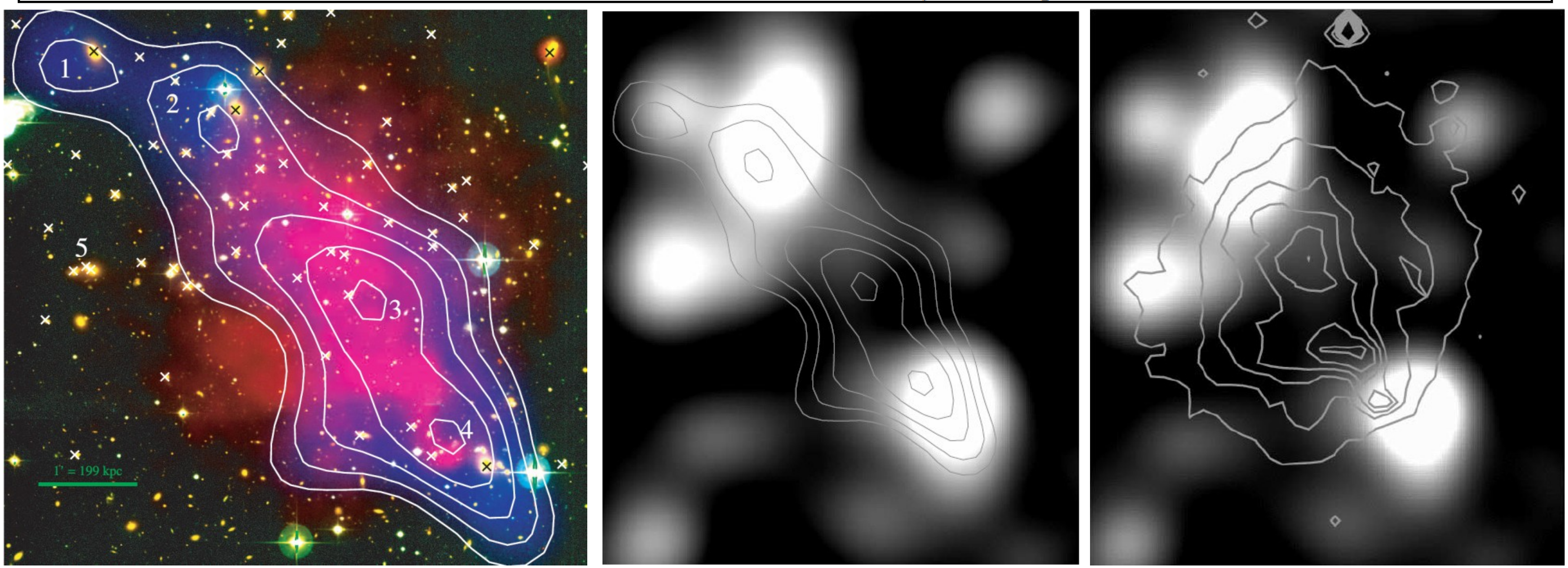
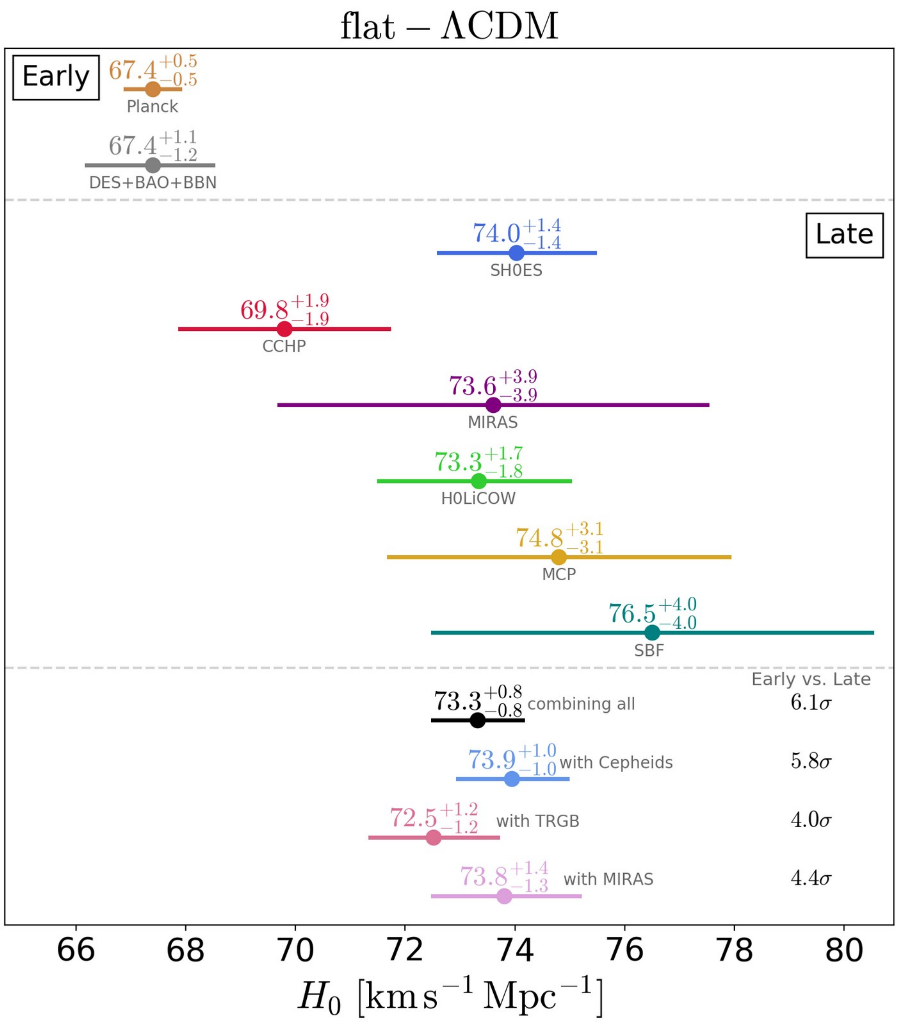


Problems with standard ΛCDM cosmological model

- **Hubble tension**: significant discrepancies between H_0 values obtained from CMBR (early universe) and SN Ia (late universe)
- SN Ia: $H_0 = 74.0 \pm 1.4$ km s^{-1} Mpc^{-1} (Riess et al. 2019, ApJ, 876, 85)
- Planck, 2018 (with 1% precision): $H_0 = 67.4 \pm 0.5$ km s^{-1} Mpc^{-1}
- 4σ - 6σ discrepancy is higher than systematic errors in the data, which therefore cannot be the explanation
- 2σ = curiosity, 3σ = tension, 4σ = discrepancy or problem, 5σ = crisis
- **Cosmological constant problem (vacuum catastrophe)**: measured values of Λ, representing vacuum energy density, are 120 orders of magnitude smaller than theoretical value of zero-point energy predicted by quantum field theory (Weinberg, S. 1989, RvMP, 61, 1)



(Verde, Treu & Riess, 2019, Nature Astronomy, 3, 891)

Exam question

1. Gravitational lenses and their cosmological applications

Literature

Textbook:

1. *Gravitational Lensing*: Strong, Weak and Micro, Book Series: Saas-Fee Advanced Courses
   - P. Schneider - *Introduction to Gravitational Lensing and Cosmology*
   - C. S. Kochanek - *Strong Gravitational Lensing*
   - P. Schneider - *Weak Gravitational Lensing*
   - J. Wambsganss - *Gravitational Microlensing*

Exercise 1

Calculate the angular Einstein radius in the case of a lensing galaxy with a mass $M = 10^{12}\, M_\odot$ at a redshift of $z_d = 0.5$ and a source at redshift $z_s = 2.0$. Assume the flat cosmological model with $H_0 = 71$ km/s/Mpc, $\Omega_M = 0.27$ and $\Omega_\Lambda = 0.73$, and use the Ned Wright's Javascript Cosmology Calculator to calculate the cosmological distances: http://www.astro.ucla.edu/~wright/CosmoCalc.html

Note that it is convenient to use the gravitational constant expressed in the following units: $G \approx 4.302 \times 10^{-3} \frac{\mathrm{pc}}{M_\odot} \frac{\mathrm{km}^2}{\mathrm{s}^2}$

Exercise 2

Estimate the mass of the lensing galaxy of Einstein Cross (Q2237+030) from angular separation of its images. Take this separation and the redshifts from CASTLES Gravitational Lens Data Base at: http://www.cfa.harvard.edu/castles/.

Assume the same cosmological model as in previous exercise.

Compare the obtained mass inside Einstein ring with the corresponding estimates given in Table 1 of Wambsganss & Paczynski, 1994, AJ, 108, 1156.

Solution 1

$D_d = 1254.5$ Mpc

$D_s = 1748.1$ Mpc

$D_{ds} = (D_{Ms} - D_{Md})/(1+z_s) = (5244.3-1881.7)/(1+2.0)$ Mpc $= 1120.9$ Mpc

$$D = \frac{D_d D_s}{D_{ds}} = 1956.4 \text{ Mpc}$$

Angular Einstein radius: $\theta_E = \sqrt{\frac{4GM}{c^2 D}} \approx 1 \times 10^{-5} \,\mathrm{rad} \approx 2''$

Note: 1 rad = $(648000 / \pi)'' \approx 206265''$

Solution 2

$z_s = 1.69,\ z_d = 0.04$

Angular separation of the images (size) $\approx 2\theta_E = 1''.78$

$\theta_E = 1''.78 / 2 = 0''.89 = (0.89 / 206265)$ rad $= 4.315 \times 10^{-6}$ rad

$D_d = 161.1$ Mpc

$D_s = 1764.8$ Mpc

$D_{ds} = (D_{Ms}\text{-}D_{Md})/(1+z_s) = (4747.3\text{-}167.5)/(1+1.69)$ Mpc $= 1702.5$ Mpc

$$D = \frac{D_d D_s}{D_{ds}} = 167\,\mathrm{Mpc}$$

$$M = \frac{c^2 D\,\theta_E^2}{4G} = 1.6 \times 10^{10} M_\odot$$

Table 1 from Wambsganss & Paczynski, 1994, AJ, 108, 1156:

$M \approx 1.5 \times 10^{10}\ M_\odot$