MASS 2023 Course: Gravitation and Cosmology

Predrag Jovanović Astronomical Observatory Belgrade

Lecture 12: gravitational lenses and their cosmological applications

- Gravitational lenses:
 - Definition and types: strong (macro and micro) and weak lensing
 - Point-like lenses: deflection angle, lens equation and Einstein radius
 - Extended lenses: surface mass density and deflection (lensing) potential
- Cosmological applications of gravitational lensing:
 - Strong lenses as natural telescopes: detection of the most distant galaxies
 - Fermat potential and lensing time delay: determination of H_0
 - Optical depth and statistics of strong lenses: constraints on the cosmological parameters
 - Microlensing applications for studying physics and spacetime geometry in vicinity of supermassive black holes
 - Weak lensing and detection of dark matter by mass reconstruction
- Problems with standard ACDM cosmological model: Hubble tension and cosmological constant problem (vacuum catastrophe)
- Exercises

Light bending in the gravitational field

- **Gravitational lens** is a massive celestial object (or a distribution of matter), located between an observer and a distant background source, which gravitational force deflects the light rays from the source, producing the multiple images of the source (macrolensing), amplification of its brightness (microlensing), or distortion of its shape (weak lensing).
- Light "falls" under the gravity
- Newton: gravitational bending of the paths of "corpuscles" (particles from which light is composed)
- General Relativity: light follows geodesics



Gravitational lens:

1. deflection is larger closer to the axis
 2. focusing to a line

Converging lens in optics: 1. deflection is larger further from the axis 2. focusing to a point





Light deflection angle





- Johann Georg von Soldner (1804) trajectory of particle with speed c: $\alpha = \frac{2GM}{c^2\xi}$
- Albert Einstein (1915) General Relativity:

$$\alpha = \frac{4GM}{c^2\xi}$$

- Eddington total solar eclipse in 1919:
 - No light bending: $\alpha = 0$ "
 - Newton's mechanics: $\alpha = 0".87$
 - GR: $\alpha = 1".75$
- Confirmation of Einstein's predictions: $\alpha_1 = 1".98 \pm 0".12$ $\alpha_2 = 1".61 \pm 0".30$





- The first identified gravitational lens: double-imaged quasar QSO 0957+561 (right), discovered by Dennis Walsh, Robert Carswell and Ray Weyman in 1979
- Both components have identical redshifts and spectra



Quasar Q2237+030 at z=1.695 (Einstein cross) and lensing galaxy ZW2237+030 at z=0.0394



Quasar RXJ1131-1231

Gravitational lensing theory

- Geometrically thin lens: the field equations of GR can be linearized if the gravitational field is weak (i.e. for the small deflection angle), and the ray can be approximated as a straight line near the deflecting mass
- Lens equation (see the figure):

$$\vec{\eta} = \frac{D_s}{D_d} \vec{\xi} - D_{ds} \vec{\hat{\alpha}}(\vec{\xi})$$

- Light deflection angle: $\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4GM}{c^2} \frac{\vec{\xi}}{\xi^2}$
- Angular coordinates $\vec{\beta}$ and $\vec{\theta}$, and scaled (reduced) deflection angle $\vec{\alpha}(\vec{\theta})$:

$$\vec{\eta} = D_s \vec{\beta}, \ \vec{\xi} = D_d \vec{\theta}, \ \vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s} \vec{\hat{\alpha}}(\vec{\theta}) \Rightarrow$$

• Dimensionless lens equation:

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

• Angular diameter distances between the observer and lens, observer and source, and lens and source: $D_d = D_A(0, z_d), D_s = D_A(0, z_s), D_{ds} = D_A(z_d, z_s)$



Einstein radius

- Solution of the lens equation for a point mass M and perfect alignment between the observer, lens and source: $\vec{\eta} = \vec{\beta} = 0 \Rightarrow$ Einstein radius
- Linear (in the lens plane): $\xi_E = \sqrt{\frac{4GM}{c^2} \frac{D_d D_{ds}}{D_s}}$

• Angular: $\theta_E = \frac{\xi_E}{D_d} = \sqrt{\frac{4GM}{c^2D}}$, where

D is effective lens distance: $D = \frac{D_d D_s}{D_{ds}}$

• **Projected** (in the source plane):

$$\eta_E = \frac{D_s}{D_d} \xi_E = \sqrt{\frac{4GM}{c^2}} \frac{D_s D_{ds}}{D_d}$$

- Typical Einstein radius:
 - for a galaxy: on the order of 1"
 - for galaxy clusters: on the order of 10"
 - for a star: on the order of μ as
- Separation between the images is twice the average Einstein radius





Point-like lenses

- Deflection angle: $\vec{\alpha}(\vec{\theta}) = \theta_E^2 \frac{\vec{\theta}}{\theta^2} \Rightarrow$
- Lens equation: $\vec{\beta} = \vec{\theta} \theta_E^2 \frac{\vec{\theta}}{\theta^2} \Rightarrow$

$$\vec{\beta} = \vec{\theta} - \theta_E^2 \frac{\vec{\theta}}{\theta^2} \middle/ \frac{\vec{\theta}}{\vec{\theta}} \Rightarrow \theta^2 - \beta \theta - \theta_E^2 = 0 \Rightarrow \frac{1}{1} \left(\frac{1}{\theta} + \frac{1}{\theta} \frac{1}{\theta} + \frac{1}{\theta} \frac{1}{\theta} \right)$$

- Image positions: $\theta_{1,2} = \frac{1}{2} \left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2} \right)$
- Point-like lenses produce 2 mirror-inverted images
- Image magnification ratio between the solid angles of the image and the source:

$$\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta} \quad \Rightarrow$$
$$\mu_{1,2} = \left(1 - \left[\frac{\theta_E}{\theta_{1,2}}\right]^4\right)^{-1}$$





Extended lenses

• 3D mass density $\rho(\vec{r})$ of an **extended lens** can be projected along the line of sight onto the lens plane to obtain the 2D **surface mass density** distribution: $\Sigma(\vec{\xi}) = \int_{0}^{D_s} \rho(\vec{r}) dz$,

where \vec{r} is a 3D vector in space, and $\vec{\xi}$ is a 2D vector in the lens plane

• Critical surface mass density is given by the lens mass M"smeared out" over the area of the Einstein ring: $\Sigma_{cr} = \frac{M}{\pi \xi_F^2} = \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}} = 0.35 \,\mathrm{g \, cm^{-2}} \left(\frac{D}{1 \,\mathrm{Gpc}}\right)^{-1}$



• A mass distribution for which $\kappa \ge 1$, i.e. $\Sigma \ge \Sigma_{cr}$ produces multiple images for some source positions (limit between "weak" from "strong" lenses)

• Deflection potential ψ : $\psi(\vec{\theta}) = \frac{1}{\pi} \int_{\mathbb{T}^2} \kappa(\vec{\theta'}) \ln |\vec{\theta} - \vec{\theta'}| d^2 \theta'$

- Deflection angle as a gradient of $\psi: \vec{\alpha} = \vec{\nabla}\psi(\vec{\theta}) \Rightarrow$ lens equation: $\left| \vec{\beta} = \vec{\theta} \vec{\nabla}\psi(\vec{\theta}) \right|$
- Poisson equation: $\nabla^2 \Phi = 4\pi G \rho \Rightarrow \nabla^2 \psi = 2\kappa$
- Different surface mass distributions ⇒ different deflection potentials ⇒ different extended lens models ⇒ different number of images (3, 4, 5, ... images of a source)



Deflection potentials of simple lens models

| Lens Model | | $\psi(heta)$ | | lpha(heta) | |
|--|-----------------|--|-------------------|---|-----------------|
| Point mass | | $\frac{D_{\rm ds}}{D_{\rm s}} \frac{4GM}{D_{\rm d}c^2} \ln \theta $ | | $\frac{D_{\rm ds}}{D_{\rm s}} \frac{4GM}{c^2 D_{\rm d} \theta }$ | |
| Singular isothermal sphere | | $rac{D_{ m ds}}{D_{ m s}}rac{4\pi\sigma^2}{c^2}\left 	heta ight $ | | $\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2}$ | |
| Softened isothermal sphere | | $\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} \left(\theta_{\rm c}^2 + \theta^2\right)^{1/2}$ | | $\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} \frac{\theta}{\left(\theta_{\rm c}^2 + \theta^2\right)^{1/2}}$ | |
| | | - | Einstein Cross | Cusp Caustic | Fold Caustic |
| | Source Plane | | | | |
| Multiple images due to lensing by a non-singular isothermal ellipsoid lens | | Image Plane | | | · (·) . |

Gravitational lensing applications

1. Strong lensing:

- by galaxies (macrolensing) multiple images of the background sources: determination of cosmological parameters (H₀ from time delays, Ω_M, Ω_Λ, Ω_k from lensing statistics)
- by stars (**microlensing**) amplification (magnification) of the background sources: detection of extrasolar planets, studying the innermost regions of active galaxies around their central supermassive black holes, constraining cosmological parameters
- by clusters of galaxies giant arcs as images of distant background galaxies: finding the most distant galaxies in the Universe (natural telescopes)

2. Weak lensing:

• by foreground matter with lower density distribution - shape distortions of the background sources: the only direct mean to detect the dark matter, studying the distribution of visible and dark matter in the Universe

1. Strong lensing by galaxy clusters (natural telescopes): finding the most distant galaxies





Finding the most distant galaxies



Distant Galaxy Lensed by Cluster Abell 2218 Hubble Space Telescope • WFPC2 • ACS

ESA, NASA, J.-P. Kneib (Caltech/Observatoire Midi-Pyrénées) and R. Ellis (Caltech) STScl-PRC04-08

Distant Gravitationally Lensed Galaxy Hubble Space Telescope Galaxy Cluster Abell 1689 ACS/WFC NICMOS Visible Light Hubble Infrared Light Hubble Infrared Light Spitzer Bouwens (UCSC), H. Ford (JHU), and G. Illingworth (UCSC) STScI-PRC08-08a NASA, ESA, and

Red arc and point: the most distant galaxy known until 2004, located at $z \sim 7 ~(\approx 13 \times 10^9 \text{ ly})$ The most distant galaxy known until 2008, located at $z \sim 7.6 ~(\approx 13 \times 10^9 \text{ ly})$

lensed galaxy image

lensed galaxy image

The most distant galaxy known until 2011, formed 13.5 billion years ago, discovered by lensing effect by galactic cluster Abell 383



MACS0647-JD: the most distant galaxy discovered in 2012: the distance to the cluster is 5.6 billion ly (z = 0.591) and to the lensed galaxy is 13.3 billion ly (z = 11)

• Current record from 2016: galaxy GN-z11 at *z* = 11.1



Four most distant galaxies ever seen detected by JWST in 2023





Gravitational lensing by galaxy cluster Abell 2744 (Pandora's Cluster)

Robertson et al. 2023, Nature Astronomy, 7, 611.

2. Fermat potential and lensing time delay

- Fermat potential: $\tau(\vec{\theta}, \vec{\beta}) = \frac{1}{2}(\vec{\theta} \vec{\beta})^2 \psi(\vec{\theta})$ is, up to an affine transformation, the **light travel time** along a ray starting at position $\vec{\beta}$, traversing the lens plane at position $\vec{\theta}$ and arriving at the observer
- Fermat principle: the physical light rays are those for which the light travel time is stationary, and lens equation is then: $\vec{\nabla} \tau(\vec{\theta}, \vec{\beta}) = 0$
- Light travel time (physical time delay function) for a lensed image:

$$\tau(\vec{\theta},\vec{\beta}) = \tau_{\text{geom}} + \tau_{\text{grav}} = \frac{D_{\Delta t}}{c} \left(\frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right), \quad D_{\Delta t} = (1 + z_d) \frac{D_d D_s}{D_{ds}},$$

where τ_{geom} is extra path length between observer and source, while τ_{grav} is retardation due to gravitational potential (Shapiro delay)

- $D_{\Delta t}$ is the **time-delay distance** which is $\propto H_0^{-1}$ and very weakly sensitive to Ω_M and Ω_{Λ}
- Lensing time delay: $\Delta t = \tau_2 \tau_1 \Rightarrow H_0$
- Estimates of H₀ depend on lens model (i.e. on deflection potential ψ)



Measuring the time delays from the light curves of the images



Fig. 9. Light curves of the two images of the QSO 0957+561A,B in two different filters. The two light curves have been shifted in time relative to each other by the measured time delay of 417 days, and in flux according to the flux ratio. The sharp drop measured in image A in Dec. 1994 and subsequently in image B in Feb. 1996 provides an accurate measurement of the time delay (data from Kundić et al. 1997)

Time delay of the circularly symmetric lenses

•
$$D_{\Delta t} = (1+z_d) \frac{D_d D_s}{D_{ds}} = \frac{c}{H_0} d_{\Delta t}$$
, where $d_{\Delta t} = \frac{d_C(z_d) \cdot d_C(z_s)}{d_C(z_s) - d_C(z_d)}$ is
dimensionless time-delay distance, and $d_C(z) = \int_0^z \frac{dz'}{\sqrt{\Omega_M (1+z')^3 + \Omega_\Lambda}}$
is dimensionless comoving distance

• Total time delay between two images is then given by (up to the first order):

$$\Delta t \approx \frac{d_{\Delta t}}{H_0} (\theta_A^2 - \theta_B^2) (1 - \langle \kappa \rangle),$$

where $<\kappa>$ is the mean surface density in the annulus between the images





(Kochanek & Schechter, 2004, astro-ph/0306040)

Determining H_0 from lensing time delays



Schechter, 2005, IAU Symposium 225, 281

3. Optical depth τ and statistics of strong lenses

- τ probability of a source being lensed (chance of seeing a lensing event), i.e. probability that at any instant of time a source is within the Einstein ring of a lens
- Cross section of strong lensing (A) area in the lens plane where the separation between the lens and source is sufficiently small for strong lensing to occur: $A = \pi \theta_E^2$
- The total τ is obtained by summing the cross sections of all deflectors between the observer and source, and it depends on cosmological parameters:

$$\tau_{SIS}(z_l, z_s, \Omega_M, \Omega_\Lambda) = \frac{1}{4\pi} \int_0^{z_s} dV \int_0^\infty d\sigma \cdot \frac{dn}{d\sigma} \cdot A_{SIS}(\sigma, \Omega_M, \Omega_\Lambda, z_l, z_s), \quad \text{where}$$

$$V = \frac{4}{3}D_C^3 \Rightarrow dV = 4D_C^2 \cdot dD_C \text{ and } A_{SIS} = 16\pi^3 \left(\frac{\sigma}{c}\right)^4 \left(\frac{D_{ls}}{D_s}\right)^2$$

- Statistical distributions obtained from differential optical depth $d\tau$: distribution per image separations $\Delta\theta \ (d\tau/d\Delta\theta)$, distribution per redshift of lens galaxies $z_l \ (d\tau/dz_l)$ and joint distribution $d^2\tau/(dz_l \ d\Delta\theta)$ per both z_l and $\Delta\theta$
- Relative probability of finding a lens at some z_l : $\delta p_l = \frac{d\tau}{dz_l}/\tau$
- Fitting a probability distribution from an observed sample of strong lenses by a modeled prediction $\Rightarrow \Omega_M \land \Omega_\Lambda$ (the results do not depend on H_0)
- It is essentially a comoving volume cosmological test

Cosmological parameters from statistics of strong lenses



4. Microlensing applications for studying physics and spacetime geometry in vicinity of SMBHs







Right: influence of a point-like microlens on X-ray radiation from a relativistic accretion disk around a SMBH



Dark matter

- Zwicky applied virial theorem to the motions of galaxies in the Coma Cluster ⇒ several hundred times more estimated than observable mass ⇒ "dunkle Materie" (Zwicky, 1933, HPA, 6, 110)
- Vera Rubin in the 1960s and 1970s: the observed rotation curves of spiral galaxies are flat ⇒ 6 times as much dark as visible mass
- DM is composed from non-baryonic particles which are so weakly interacting that they move purely under the influence of gravity ⇒ it can be directly detected only by weak lensing
- <u>Hypothesis</u>: a spherical **dark matter halo** around a spiral galaxy (Navarro, Frenk & White, 1996, ApJ, 462, 563):





5. Weak lensing by dark matter



Gravitational lensing by a massive cluster of galaxies with two different distributions of the same amount of the dark matter over the cluster (orange), causing a particular distortion of the background galaxies (white and blue).

Detection of dark matter by weak lensing

- An observed galaxy ellipticity is a combination of its intrinsic ellipticity and shear
- Shear can be estimated by averaging over many galaxy images, assuming that the intrinsic ellipticities are *randomly oriented*
- Thus, weak lensing can be used for measuring the local shear γ
- Both, shear γ and surface mass density κ are second partial derivatives of the deflection potential $\psi \Rightarrow$ they are linearly related
- Therefore, it is possible to derive the surface mass density κ from the measured values of shear γ
- This technique is called **mass reconstruction** and is usually used in the case of clusters of galaxies



Local averages of ellipticities of background galaxies ⇒ local shear estimates (green sticks) ⇒ surface mass density mass map



< 0

>0

Mass reconstructions by weak lensing



FIG. 3.—Full 2048 \times 2048 pixel *I*-band CCD image of MS 1054-03 with the ellipses drawn around all the 2395 objects in the *I* > 21.5 catalog

FIG. 5.—Contour plot of the surface mass density (black contour lines) and cluster light distribution (white contour lines) overlaid on the 2048² pixel optical image of the cluster.

Left: image of cluster of galaxies MS1054–03 with about 2400 measured galaxy ellipticities Right: mass reconstruction (black), compared to light distribution (white)

Detection of dark matter in the case of "Bullet Cluster" (1E 0657-558)

THE ASTROPHYSICAL JOURNAL, 648:L109–L113, 2006 September 10 © 2006. The American Astronomical Society. All rights reserved. Printed in U.S.A.

A DIRECT EMPIRICAL PROOF OF THE EXISTENCE OF DARK MATTER¹

Douglas Clowe,² Maruša Bradač,³ Anthony H. Gonzalez,⁴ Maxim Markevitch,^{5,6} SCOTT W. RANDALL,⁵ CHRISTINE JONES,⁵ AND DENNIS ZARITSKY²

Received 2006 June 6; accepted 2006 August 3; published 2006 August 30



Counterexample: a dark core in Abell 520



THE ASTROPHYSICAL JOURNAL, 668:806-814, 2007 October 20 © 2007. The American Astronomical Society. All rights reserved. Printed in U.S.A.

A DARK CORE IN ABELL 520¹

ANDISHEH MAHDAVI, HENK HOEKSTRA, ARIF BABUL, AND DAVID D. BALAM Department of Physics and Astronomy, University of Victoria, Victoria, BC V8W 3P6, Canada

AND

PETER L. CAPAK California Institute of Technology, MC 105-24, 1200 East California Boulevard, Pasadena, CA 91125 Received 2007 February 10; accepted 2007 June 18

THE ASTROPHYSICAL JOURNAL, 718:60–65, 2010 July 20 © 2010. The American Astronomical Society. All rights reserved. Printed in the U.S.A.

BULLET CLUSTER: A CHALLENGE TO ACDM COSMOLOGY

JOUNGHUN LEE¹ AND EIICHIRO KOMATSU²

 Bullet cluster: high infall velocities of subclusters around massive main clusters are incompatible with their ACDM predictions

Problems with standard ΛCDM model

- **1.Hubble tension**: significant discrepancies between H_0 values obtained from CMBR (early universe) and SN Ia (late universe)
- SN Ia: $H_0 = 74.0 \pm 1.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ (Riess et al. 2019, ApJ, 876, 85)
- Planck, 2018 (with 1% precision): $H_0 = 67.4 \pm 0.5 \text{ km s}^{-1} \text{ Mpc}^{-1}$
- Discrepancy > 5σ and is higher than systematic errors in the data
- 2σ = curiosity, 3σ = tension, 4σ = discrepancy or problem, 5σ = crisis



2.Cosmological constant problem (vacuum catastrophe): measured values of Λ, representing vacuum energy density, are 120 orders of magnitude smaller than theoretical value of zero-point energy predicted by quantum field theory (Weinberg, S. 1989, RvMP, 61, 1)

Exam question

1. Gravitational lenses and their cosmological applications

Literature

Textbook:

- 1. Gravitational Lensing: Strong, Weak and Micro, Book Series: Saas-Fee Advanced Courses
 - P. Schneider Introduction to Gravitational Lensing and Cosmology
 - C. S. Kochanek Strong Gravitational Lensing
 - P. Schneider Weak Gravitational Lensing
 - J. Wambsganss Gravitational Microlensing

Exercise 1

Estimate the mass of the lensing galaxy of Einstein Cross (Q2237+030) from angular separation of its images. Take this separation and the redshifts from CASTLES Gravitational Lens Data Base at: <u>http://www.cfa.harvard.edu/castles/</u>.

Assume the flat cosmological model with $H_0 = 71$ km/s/Mpc, $\Omega_M = 0.27$ and $\Omega_\Lambda = 0.73$, and use the Ned Wright's Javascript Cosmology Calculator to calculate the cosmological distances: <u>http://www.astro.ucla.edu/~wright/CosmoCalc.html</u>

Note that it is convenient to use the gravitational constant expressed in the following units: $G \approx 4.302 \times 10^{-3} \frac{\text{pc}}{M_{\odot}} \frac{\text{km}^2}{\text{s}^2}$

Compare the obtained mass inside Einstein ring with the corresponding estimates given in Table 1 of Wambsganss & Paczynski, 1994, AJ, 108, 1156.

Exercise 2

Measured time delay between 2 images of gravitational lens system HE2149-2745 is $\Delta t = 103$ days, positions of the images in respect to the lens galaxy are: $x_A = 0$ ".714, $y_A = -1$ ".150, and $x_B = -0$ ".176, $y_B = 0$ ".296, and redshifts of the source and lens are: $z_s = 2.03$ and $z_d = 0.50$. Estimate the value of H_0 , assuming the lens model with $\langle \kappa \rangle = 0.22$ (Kochanek, 2002, ApJ, 578, 25), as well as the following cosmological model: $\Omega_M = 0$ and $\Omega_{\Lambda} = 1$.

Solution 1

 $z_{\rm s} = 1.69, z_{\rm d} = 0.04$ Angular separation of the images (size) $\approx 2\theta_F = 1''.78$ $\theta_E = 1''.78 / 2 = 0''.89 = (0.89 / 206265) \text{ rad} = 4.315 \text{ x } 10^{-6} \text{ rad}$ **Note:** 1 rad = $(648000 / \pi)'' \approx 206265''$ $D_{d} = 161.1 \text{ Mpc}$ $D_{\rm s} = 1764.8 \; {\rm Mpc}$ $D_{ds} = (D_{Ms} - D_{Md})/(1 + z_s) = (4747.3 - 167.5)/(1 + 1.69) \text{ Mpc} = 1702.5 \text{ Mpc}$ $D = \frac{D_d D_s}{D_{ds}} = 167 \,\text{Mpc}$ Angular Einstein radius $\Rightarrow M = \frac{c^2 D \,\theta_E^2}{4G} = 1.6 \times 10^{10} M_{\odot}$

Table 1 from Wambsganss & Paczynski, 1994, AJ, 108, 1156: $M \approx 1.5 \times 10^{10} M_{\odot}$

Solution 2

$$\begin{aligned} \theta_A &= 1''.354 = 6.564 \times 10^{-6} \text{rad} \\ \theta_B &= 0''.344 = 1.668 \times 10^{-6} \text{rad} \end{aligned} \right\} \Rightarrow \theta_A^2 - \theta_B^2 = 4.03 \times 10^{-11} \\ \Omega_M &= 0 \land \Omega_\Lambda = 1 \Rightarrow d_C(z) = \int_0^z \frac{dz'}{\sqrt{\Omega_M (1 + z')^3 + \Omega_\Lambda}} = z \Rightarrow \\ d_{\Delta t} &= \frac{d_C(z_d) \cdot d_C(z_s)}{d_C(z_s) - d_C(z_d)} = \frac{z_d z_s}{z_s - z_d} \\ \Delta t &\approx \frac{d_{\Delta t}}{H_0} (\theta_A^2 - \theta_B^2) (1 - \langle \kappa \rangle) \Rightarrow \quad H_0 = \frac{z_d z_s (\theta_A^2 - \theta_B^2) (1 - \langle \kappa \rangle)}{(z_s - z_d) \Delta t} \\ H_0 &= \frac{0.50 \cdot 2.03 \cdot 4.03 \times 10^{-11} \cdot 0.78}{z_s - z_d} + \frac{3.0856776 \times 10^{19} \,\text{km}}{z_s - z_s - z_s} = 72.3 \text{ km} \end{aligned}$$

 $H_0 = \frac{0.50 \cdot 2.03 \cdot 4.03 \times 10^{-11} \cdot 0.78}{(2.03 - 0.50) \cdot 103 \cdot 86400 \,\mathrm{s}} \cdot \frac{3.0856776 \times 10^{15} \,\mathrm{km}}{\mathrm{Mpc}} = 72.3 \,\frac{\mathrm{km}}{\mathrm{s} \cdot \mathrm{Mpc}}$

• Note: $1 \text{ Mpc} = 3.0856776 \times 10^{19} \text{ km}$