

MASS 2026 Course:
Gravitation and Cosmology

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Lecture 01

- Brief overview of Newton's theory of gravity
- General Relativity (GR) as a geometric theory of gravitation
- Gravitational field
- Concept of spacetime: events and worldlines
- Manifolds
- Spacetime as 4-dimensional pseudo-Riemannian manifold
- Some important notation and conventions:
 - Einstein summation convention
 - General coordinate transformations
- Scalars, vectors and dual vectors
- Contravariant and covariant transformation rules
- Exercises

Newton's theory of gravity

- The gravitational force between two objects of masses M and m separated by a vector \vec{r} : $\vec{F}(\vec{r}) = -\frac{GMm}{r^2}\hat{r}$
- The force acts on particle of mass m and gives it an acceleration according to Newton's second law: $\vec{F}(\vec{r}) = m\vec{g}(\vec{r})$
- Gravitational field is described by gravitational acceleration: $\vec{g}(\vec{r}) = -\frac{GM}{r^2}\hat{r}$
- Equivalently, the gravitational potential Φ is related to the mass density ρ by Poisson's equation: $\nabla^2\Phi = 4\pi G\rho$
- The acceleration is then given by the gradient of the potential: $\vec{g}(\vec{r}) = -\nabla\Phi(\vec{r})$
- In the case of point mass M , Newton's gravitational potential is: $\Phi(r) = -\frac{GM}{r}$
- A reminder of the fundamental vector calculus operators:

- Gradient: $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$

- Divergence of a vector $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$: $\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

- Laplace operator: $\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$ (divergence $\nabla \cdot$ of gradient ∇)

General Relativity (GR)

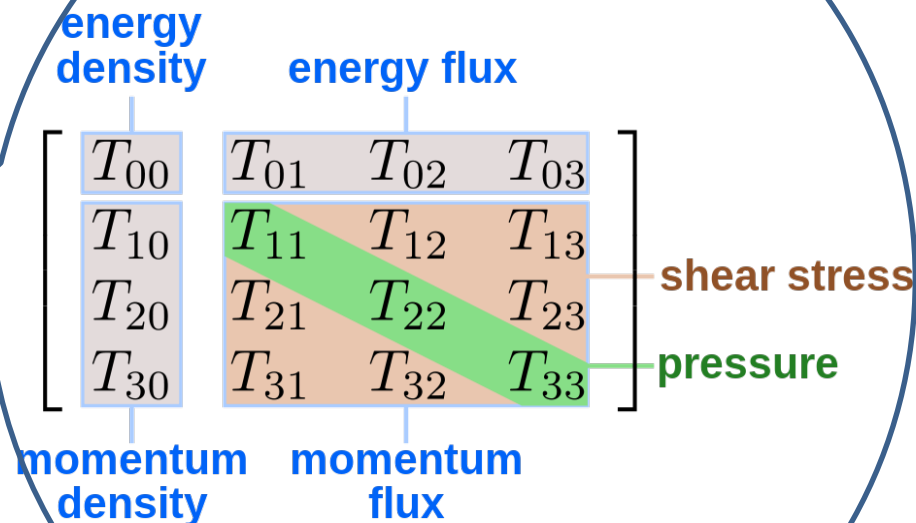
- GR is the standard theory of gravity, formulated by Albert Einstein in 1915
- It describes gravitation in terms of differential geometry of curved spacetime
- Fundamental interaction of gravitation as a result of spacetime being curved by matter and energy is described by field equations of GR:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

where:

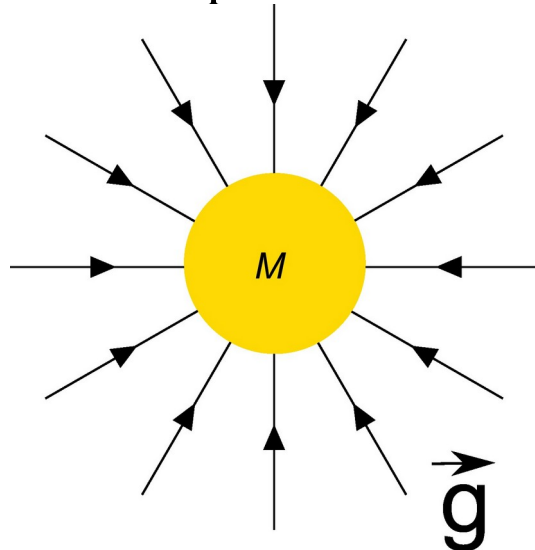
- $R_{\mu\nu}$ - Ricci curvature tensor,
- $g_{\mu\nu}$ - metric tensor,
- $T_{\mu\nu}$ - stress-energy tensor
- R - Ricci curvature scalar,
- G - Newton's gravitational constant,
- c - speed of light in vacuum
- Einstein tensor: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$

Source of gravity:

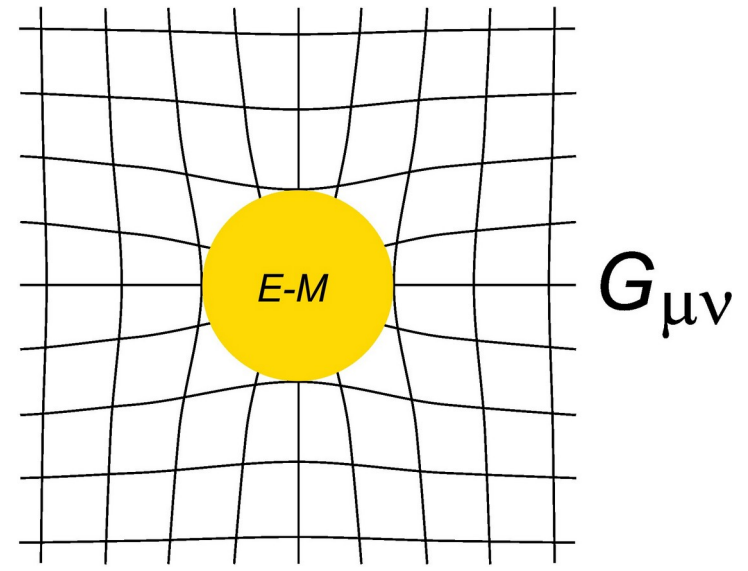


Gravitational field

- In physics, a field is a physical quantity, represented by a scalar, vector or tensor, that has a value for each point in space and time
- A function assigning a scalar, vector, or tensor to each point in space and time:
 - scalar fields (temperature, humidity, pressure, ...)
 - vector fields (Newton's gravitational field, electric field, ...)
 - tensor fields (stress and strain in materials, Riemann curvature tensor, ...)
- Gravitational field in Newton's theory of gravity is a vector field that exists in the space around every mass
- In contrast, gravity in General Relativity (GR) is not due to an effective force field, but a result of spacetime curvature



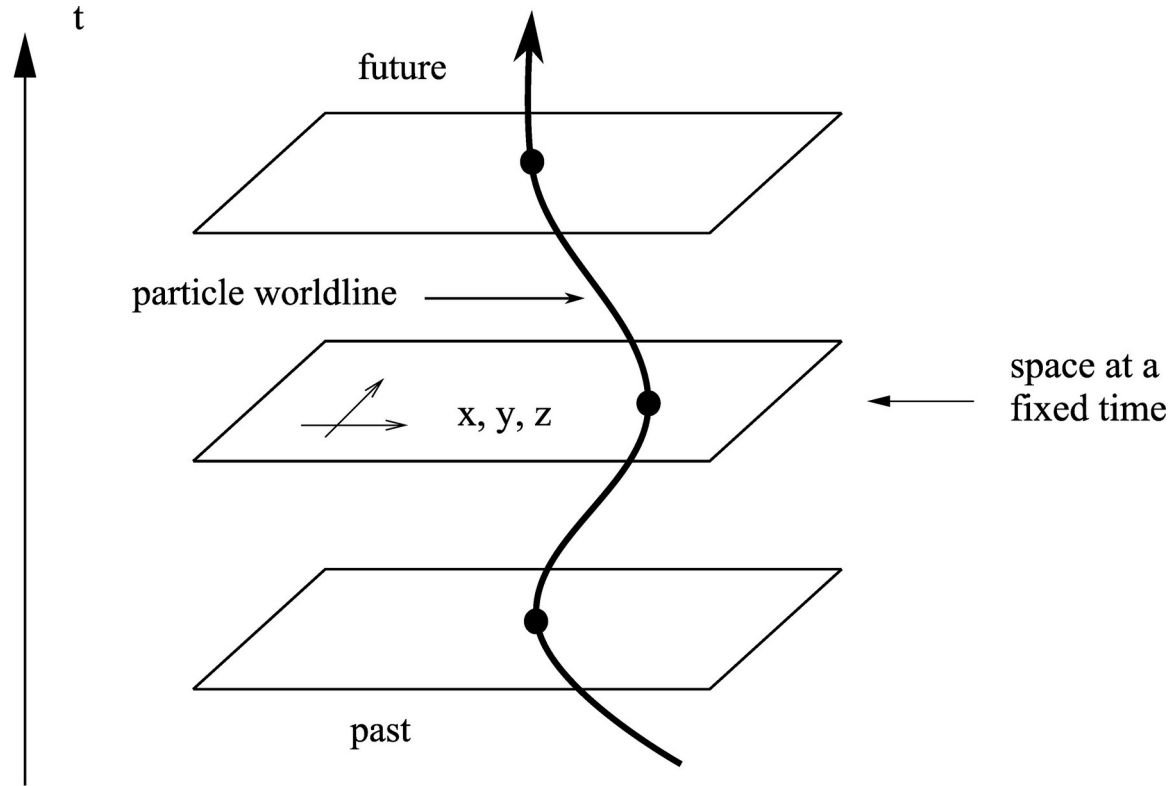
In Newton's theory of gravity, mass is the source of a gravitational field \vec{g}



In GR, mass-energy curves spacetime (Einstein tensor $G_{\mu\nu}$)

Concept of spacetime

- **Spacetime** is a 4-dimensional set with elements labeled by 3 dimensions of space and 1 of time
- **Event** is an individual point in spacetime
- **Worldline** is a curve through spacetime (a parametrized 1-dimensional set of events), representing the path of a particle
- In Newtonian mechanics there is:
 - no limit on particle velocity
 - absolute division of spacetime into well-defined slices of space at a fixed moment in time
 - notion of **absolute simultaneity** of two events which occur at the same time
 - **absolute space**: the spatial interval between two simultaneous events is observer independent
 - **absolute time**: time interval between two events has absolute significance and all observers agree on its value
- **Euclidean geometry**: space interval Δl between two simultaneous events is given by Pythagorean theorem:
$$\Delta l^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$



Some important notation and conventions

- From now on we will use the following notation and convention:
 - Greek indices (α, β, \dots) run from 0 to 3
 - Lower-case latin indices (a, b, \dots) run from 1 to 3
 - **Einstein summation convention:** indices which appear both as superscripts and subscripts are summed over

- Example: $y = \sum_{i=1}^3 c_i x^i = c_1 x^1 + c_2 x^2 + c_3 x^3$ is simplified to: $y = c_i x^i$

$$x^0 = ct$$

$$x^1 = x$$

$$x^2 = y$$

$$x^3 = z$$

- The superscripts are not exponents but are indices of coordinates on spacetime, with 0 denoting the time coordinate: x^μ :

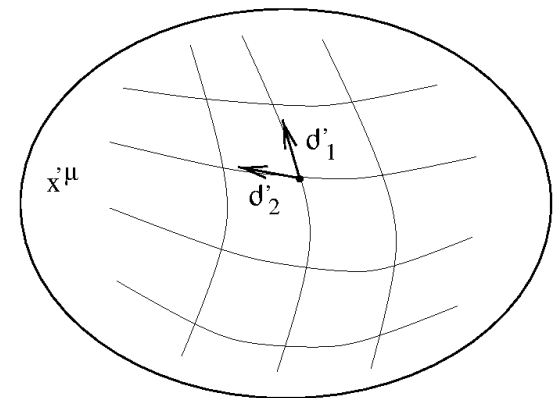
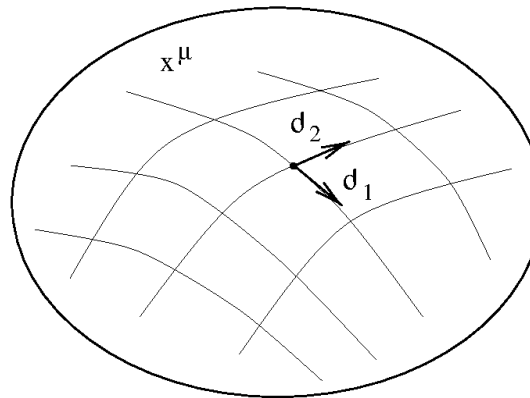
- **General coordinate transformations**

(which must be invertible) and their

inverse transformations:

$$x^\mu \rightarrow x'^\mu = x'^\mu(x^0, x^1, x^2, x^3)$$

$$x'^\mu \rightarrow x^\mu = x^\mu(x'^0, x'^1, x'^2, x'^3)$$



- A quantity is **invariant** if its value is unchanged under coordinate transformations
- An equation is **covariant** if its form is unchanged under coordinate transformations

GR as a geometric theory of gravity

Physics	Geometry
<ul style="list-style-type: none">• When and the where of a physical phenomenon constitute an event.• The set of all events is a continuous space, named spacetime• Gravitational phenomena are manifestations of the geometry of spacetime• Point-like particles move in spacetime following special world-lines that are “straight”• The laws of physics are the same for all observers	<ul style="list-style-type: none">• An event is a point in a topological space• Spacetime is a differentiable manifold M• The gravitational field is a metric g on M• Straight lines are geodesics• Field equations are generally covariant

- Newtonian gravity vs. GR:

Spacetime = Differentiable Manifold

Gravitational Potentials = Metric

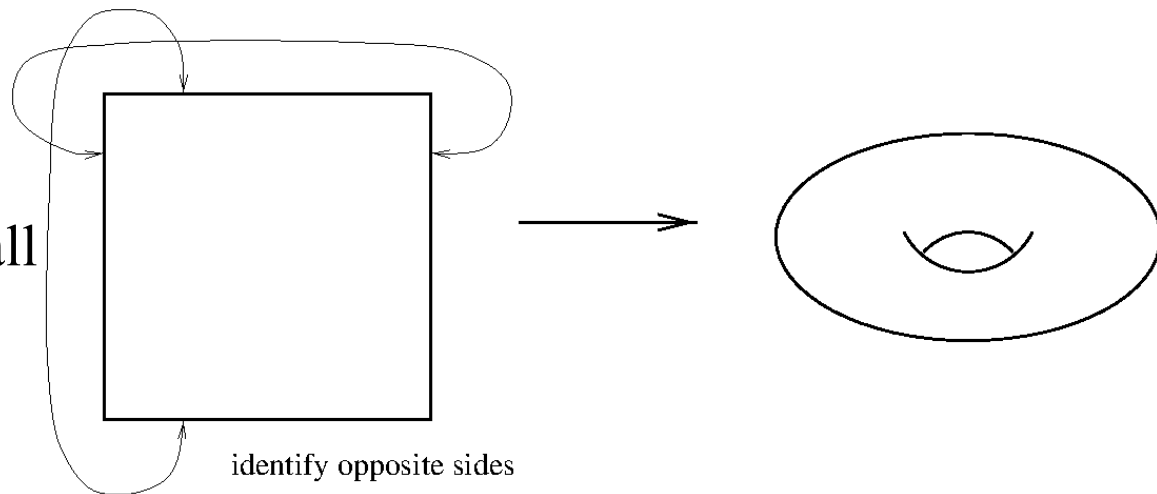
Forces = Connections

Manifolds

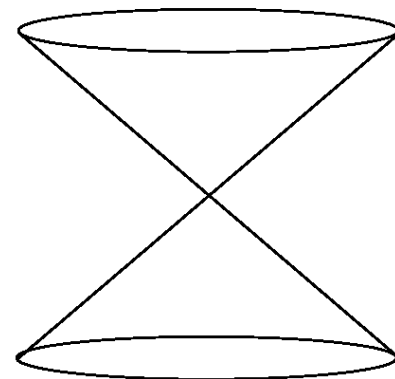
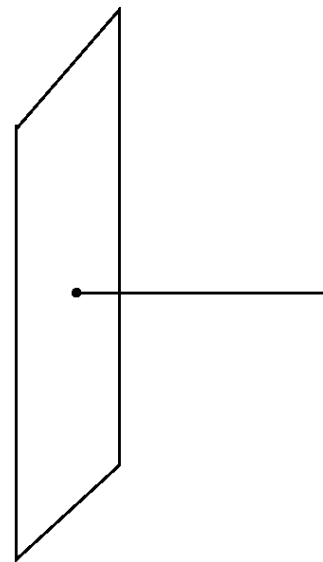
- **Manifold** (or differentiable manifold) is a topological space which may be curved and have a complicated topology, but locally looks like Euclidean space \mathbf{R}^n near each point
- Manifold is constructed by smoothly joining together the Euclidean-like local regions

- Examples of manifolds:

- Euclidean space \mathbf{R}^n
- n -sphere S^n , since it is the locus of all points at some fixed distance from the origin in \mathbf{R}^{n+1}
- The n -torus T^n , since it can be constructed from an n -dimensional cube by identifying its opposite sides



- These are not manifolds because somewhere they do not look locally like \mathbf{R}^n :
 - one-dimensional line running into a two-dimensional plane
 - two cones stuck together at their vertices



Spacetime as 4-dimensional manifold

- **Metric** is a function that defines the distance between two events within spacetime, satisfying axioms of non-negativity, symmetry, and the triangle inequality
- An example is Euclidean metric d that defines the distance between two points in Euclidean space \mathbf{R}^3 : $d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- **Metric tensor** is a symmetric 4x4 matrix defining spacetime geometry, distance and curvature
- Two conventions for metric signature (the signs of eigenvalues in the metric tensor):
(-, +, +, +) or (+, -, -, -)
- **Flat spacetime** is described by the constant **Minkowski metric**, i.e. with metric tensor:
 $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$
- **Curved spacetime** is described by a variable **Riemannian metric**, i.e. with metric tensor $g_{\mu\nu}$ that varies from point to point
- GR is a metric theory of gravitation in which 4-dimensional **pseudo-Riemannian manifold** M , equipped with a **Riemannian metric** g , is representing spacetime: pair (M, g) is the mathematical model of spacetime in GR
- Pseudo-Riemannian manifold is a differentiable manifold with a **metric tensor** $g_{\mu\nu}$ that is everywhere non-degenerate, meaning that the determinant $g = |g_{\mu\nu}|$ doesn't vanish and thus $g_{\mu\nu}$ is invertible
- Spacetime interval ds between two events with a given infinitesimal coordinate separation is determined by the metric tensor $g_{\mu\nu}$: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
- ds^2 is invariant under general coordinate transformations:

$$ds'^2 = ds^2 \quad \Leftrightarrow \quad g'_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} dx^\mu dx^\nu$$

Scalars, vectors and dual vectors

- Formulating equations which are invariant under general coordinate transformations requires knowledge about how certain quantities behave under such transformations
- **Scalars, vectors** and **tensors** are objects with particularly simple transformation properties under general coordinate transformations
- The simplest transformation rule is that of the **scalars**, which do not change under general coordinate transformations (e.g. numbers and metric function ds)
- The next simplest transformation rule is that of a **vector** which in a curved spacetime, in contrast to a flat space where it is stretching from one point to another, is an object associated with (or located at) a single point
- Vectors in four-dimensional spacetime are referred to as **four-vectors**
- Any vector A can be written as a linear combination of its basis vectors $\hat{e}_{(\mu)}$ as: $A = A^\mu \hat{e}_{(\mu)}$, where the coefficients A^μ are the **components** of the vector A
- Often, the basis is entirely omitted and shorthand "the vector A^μ " is used (i.e. vectors in GR are labeled just with upper indices)
- **Dual vectors, covectors** or **one-forms** are linear, scalar-valued functions of ordinary vectors which also transform according to an equally simple rule
- Components of a dual vector are labeled with lower indices: $\omega = \omega_\mu \hat{\theta}^{(\mu)}$
- A dual vector ω acts on a vector V , and vice versa, to produce a real number (scalar):

$$V(\omega) \equiv \omega(V) = \omega_\mu V^\mu \in \mathbf{R}$$

Contravariant and covariant transformation rules

- **Contravariant vectors** are vectors with upper indices
- **Covariant vectors** are dual vectors with lower indices
- Example: for a n -component column contravariant vector, the corresponding covariant vector is a n -component row vector and the action is ordinary matrix multiplication

$$V = \begin{pmatrix} V^1 \\ V^2 \\ \vdots \\ V^n \end{pmatrix}, \quad \omega = (\omega_1 \quad \omega_2 \quad \cdots \quad \omega_n),$$

$$\omega(V) = (\omega_1 \quad \omega_2 \quad \cdots \quad \omega_n) \begin{pmatrix} V^1 \\ V^2 \\ \vdots \\ V^n \end{pmatrix} = \omega_i V^i$$

- Under a coordinate transformation $x^\mu \rightarrow x'^\mu$, contravariant vectors transform

according to **contravariant transformation rule** into:

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

- Under a coordinate transformation $x^\mu \rightarrow x'^\mu$, covariant vectors transform according

to **covariant transformation rule** into:

$$U'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} U_\nu$$

- Examples of contravariant and covariant transformation rules:

1. The coordinate differential is a contravariant vector: $dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$

2. Gradient of a scalar field ϕ is a covariant vector: $\frac{\partial \phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \phi}{\partial x^\nu}$

Exam questions

1. Spacetime as 4-dimensional pseudo-Riemannian manifold
2. Scalars, vectors, dual vectors, contravariant and covariant transformation rules

Literature

- Textbook: Weinberg, S., 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, Wiley-VCH

Exercise 1

- Consider two coordinate systems in a 2-dimensional space: a Cartesian one where $x^\mu = (x, y)$ and a polar one where $x'^\mu = (r, \theta)$. Find the coordinate transformation $x^\mu \rightarrow x'^\mu$ and inverse coordinate transformation $x'^\mu \rightarrow x^\mu$.

Exercise 2

- Write down $X_{ij}Y^{ij}$ explicitly as a sum of terms in $n = 2$ and $n = 3$ dimensions, respectively

Exercise 3

- Write down $X_i Y^{ij} X_j$ explicitly as a sum of terms in $n = 2$ and $n = 3$ dimensions, respectively, assuming the following symmetry: $Y^{ij} = Y^{ji}$

Exercise 4

- Given a vector $V = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and a dual vector $\alpha = [1 \quad 4 \quad 0]$, find the scalar result of the action $\alpha(V)$.

Solution 1

$$x^\mu \rightarrow x'^\mu : \begin{cases} r = (x^2 + y^2)^{1/2} \\ \theta = \arctan(y/x) \end{cases}$$

$$x'^\mu \rightarrow x^\mu : \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

Solution 2

• In $n = 2$ dimensions: $X_{ij}Y^{ij} = X_{11}Y^{11} + X_{12}Y^{12} + X_{21}Y^{21} + X_{22}Y^{22}$

• In $n = 3$ dimensions:

$$\begin{aligned} X_{ij}Y^{ij} &= X_{11}Y^{11} + X_{12}Y^{12} + X_{13}Y^{13} + X_{21}Y^{21} + X_{22}Y^{22} + \\ &+ X_{23}Y^{23} + X_{31}Y^{31} + X_{32}Y^{32} + X_{33}Y^{33} \end{aligned}$$

Solution 3

- In $n = 2$ dimensions: $X_i Y^{ij} X_j = Y^{11} (X_1)^2 + 2Y^{12} X_1 X_2 + Y^{22} (X_2)^2$

- In $n = 3$ dimensions:

$$\begin{aligned} X_i Y^{ij} X_j &= Y^{11} (X_1)^2 + 2Y^{12} X_1 X_2 + 2Y^{13} X_1 X_3 + Y^{22} (X_2)^2 + \\ &+ 2Y^{23} X_2 X_3 + Y^{33} (X_3)^2 \end{aligned}$$

Solution 4

$$\alpha(V) = [1 \quad 4 \quad 0] \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 1 \cdot 2 + 4 \cdot (-1) + 0 \cdot 3 = 2 - 4 + 0 = -2$$