MASS 2026 Course:
Gravitation and Cosmology

Predrag Jovanović
Astronomical Observatory Belgrade

Lecture 02

- Tensors, their direct product and their transformation rules
- Metric (fundamental) tensor
- Tensor algebra:
  - Direct product
  - Linear combination
  - Contraction and trace
  - Raising and lowering indices
  - Scalar product of two vectors
  - Inner product
- Symmetric and antisymmetric tensors
- Exercises

Tensors

- **Contravariant vector** transformation rule: $$\boxed{V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}}$$
- **Covariant vector** transformation rule: $$\boxed{U'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} U_{\nu}}$$
- **Tensor** is a straightforward generalization of vectors and dual vectors
- Tensor T with upper indices $\mu, \nu, \ldots$ and lower indices $\kappa, \lambda, \ldots$ transforms under a coordinate transformation $x^{\mu} \to x'^{\mu}$ like the product of contravariant vectors $U^{\mu} W^{\nu} \cdots$ and covariant vectors $V_{\kappa} Y_{\lambda} \cdots$
- Example: $T'^{\mu}{}_{\nu}{}^{\lambda} = A'^{\mu} B'_{\nu} C'^{\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x^{\beta}}{\partial x'^{\nu}} B_{\beta} \frac{\partial x'^{\lambda}}{\partial x^{\gamma}} C^{\gamma} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x'^{\lambda}}{\partial x^{\gamma}} T^{\alpha}{}_{\beta}{}^{\gamma}$
- Tensor T of **type** (k, l) has k contravariant (upper) indices and l covariant (lower) indices
- A tensor is of **mixed type** if neither of k and l is 0
- **Rank** (or o**rder**) of a tensor is the total number of indices: $k + l$
- Tensors may have an arbitrary number of indices
- An n^{th}-rank tensor in m-dimensional space is a mathematical object that has n indices and m^{n} components

Tensor transformation rules

- Tensor components obey the following transformation rules:

1) Zeroth-rank or type (0, 0) tensors or **scalars** are functions which remain invariant under the coordinate transformations: $\phi' = \phi'(x'^\mu) = \phi(x^\mu) = \phi$ (e.g. metric *ds*^{2})

2) First-rank tensors (vectors) of type (1, 0): $T'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} T^\nu$ (contravariant vectors)

   and of type (0, 1): $T'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu} T_\nu$ (covariant vectors)

3) Second-rank tensors of type (2, 0): $T'^{\mu\nu} = \dfrac{\partial x'^\mu}{\partial x^\rho} \dfrac{\partial x'^\nu}{\partial x^\sigma} T^{\rho\sigma}$,

   of type (0, 2): $T'_{\mu\nu} = \dfrac{\partial x^\rho}{\partial x'^\mu} \dfrac{\partial x^\sigma}{\partial x'^\nu} T_{\rho\sigma}$,

   of type (1, 1): $T'^\mu{}_\nu = \dfrac{\partial x'^\mu}{\partial x^\rho} \dfrac{\partial x^\sigma}{\partial x'^\nu} T^\rho{}_\sigma$ (mixed tensor of rank 2)

- General tensor of type (*n*, *m*): $T'^{\alpha\ldots\beta}{}_{\gamma\ldots\delta} = \dfrac{\partial x'^\alpha}{\partial x^\alpha} \cdots \dfrac{\partial x'^\beta}{\partial x^\beta} \dfrac{\partial x^\gamma}{\partial x'^\gamma} \cdots \dfrac{\partial x^\delta}{\partial x'^\delta} T^{\alpha\ldots\beta}{}_{\gamma\ldots\delta}$

- Indices which are not summed over are called **free indices**, while indices which are summed over are called **dummy indices**

Metric (fundamental) tensor

- **Covariant metric tensor** is a symmetric type (0, 2) tensor: $g_{\mu\nu} = g_{\nu\mu}$, and thus it has 10 independent components
- Spacetime interval *ds* between two events with infinitesimal coordinate separation:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

- The most important roles of metric tensor $g_{\mu\nu}$ in GR:
  - it supplies a notion of "past"' and "future"
  - it allows the computation of path length and proper time;
  - it determines the "shortest distance" between two points (and therefore the motion of test particles)
  - it replaces the Newtonian gravitational field
  - it provides a notion of locally inertial frames
  - it determines causality
  - it replaces the Euclidean three-dimensional dot product of Newtonian mechanics
- **Contravariant metric tensor** $g^{\lambda\mu}$ is the inverse of $g_{\mu\nu}$, so that $g^{\lambda\mu} g_{\mu\nu} = \delta^\lambda_\nu$,

  where δ^μ_ν is the **Kronecker delta symbol**: $\delta^\mu_\nu = \begin{cases} 0 & (\nu \neq \mu) \\ 1 & (\nu = \mu) \end{cases}$
- δ^μ_ν is a mixed tensor, and aside from the scalars and zero (i.e. a tensor with all components equal to zero), δ^μ_ν (together with its direct products) is the only tensor whose components are the same in all coordinate systems

Tensor algebra I

- **Direct product**: the product of the components of two tensors yields a tensor whose upper and lower indices consist of all the upper and lower indices of the two original tensors, and whose rank is equal to the sum of the two original ranks
- Direct product of two tensors is sometimes denoted by sign $\otimes$, but it is also common to simply write the tensors adjacent to each other and omit the $\otimes$ sign
  - Example 1: if $A^{\mu}{}_{\nu}$ and B^{ρ} are tensors, then $T^{\mu}{}_{\nu}{}^{\rho} \equiv A^{\mu}{}_{\nu} B^{\rho}$ is also a tensor
  - Example 2: direct product of two tensors *A* and *B* in matrix notation is:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1q} & \cdots & \cdots & a_{1n}b_{11} & a_{1n}b_{12} & \cdots & a_{1n}b_{1q} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{11}b_{2q} & \cdots & \cdots & a_{1n}b_{21} & a_{1n}b_{22} & \cdots & a_{1n}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & a_{11}b_{p2} & \cdots & a_{11}b_{pq} & \cdots & \cdots & a_{1n}b_{p1} & a_{1n}b_{p2} & \cdots & a_{1n}b_{pq} \\ \vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1q} & \cdots & \cdots & a_{mn}b_{11} & a_{mn}b_{12} & \cdots & a_{mn}b_{1q} \\ a_{m1}b_{21} & a_{m1}b_{22} & \cdots & a_{m1}b_{2q} & \cdots & \cdots & a_{mn}b_{21} & a_{mn}b_{22} & \cdots & a_{mn}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1} & a_{m1}b_{p2} & \cdots & a_{m1}b_{pq} & \cdots & \cdots & a_{mn}b_{p1} & a_{mn}b_{p2} & \cdots & a_{mn}b_{pq} \end{bmatrix}$$

Tensor algebra II

- **Linear combination:** a linear combination of tensors with the same upper and lower indices is a tensor with these indices
- Example: let ${A^\mu}_\nu$ and ${B^\mu}_\nu$ be mixed tensors, and let ${T^\mu}_\nu \equiv a{A^\mu}_\nu + b{B^\mu}_\nu$, where a and b are scalars; then ${T^\mu}_\nu$ is also a tensor
- **Contraction**: Setting an upper and lower index equal and summing it over its four values yields a new tensor with these two indices absent and with rank reduced by 2
  - Example: if ${T^\mu}_\nu{}^{\rho\sigma}$ is a tensor, then $T^{\mu\rho} \equiv {T^\mu}_\nu{}^{\rho\nu}$ is also a tensor
- Tensor contraction is a generalization of **trace** in the sense that the trace is the simplest type of tensor contraction, namely a second-rank tensor contraction:
  - Example: trace of a mixed second-rank tensor ${T^\mu}_\nu$ is a scalar: $T = \mathrm{tr}\,{T^\mu}_\nu = {T^\mu}_\mu$
- **Raising and lowering indices** is an operation obtained by combining the previous three operations

Tensor algebra III

- **Lowering indices**: if we take the direct product of a contravariant or mixed tensor *T* with the metric tensor $g_{\mu\nu}$, and contract the index μ with one of the contravariant indices of *T*, we get a new tensor in which this contravariant index is replaced by a covariant index ν
- Example: if $T^{\mu\rho}{}_{\sigma}$ is a tensor then $S_{\nu}{}^{\rho}{}_{\sigma} \equiv g_{\mu\nu} T^{\mu\rho}{}_{\sigma}$ is also a tensor
- **Raising indices**: if we take the direct product of a covariant or mixed tensor *T* with the inverse metric tensor $g^{\mu\nu}$, and contract the index μ with one of the covariant indices of *T*, we get a new tensor in which this covariant index is replaced by a contravariant index ν
- Example: if $S_{\mu}{}^{\rho}{}_{\sigma}$ is a tensor then $R^{\nu\rho}{}_{\sigma} \equiv g^{\mu\nu} S_{\mu}{}^{\rho}{}_{\sigma}$ is also a tensor
- Lowering an index and then raising it again gives back the original tensor
- Tensors obtained by raising and lowering indices are called **associated tensors** and are physically equivalent
- Tensor obtained by raising one index on the metric tensor $g_{\mu\nu}$ or by lowering one index on the inverse metric tensor $g^{\mu\nu}$, is the **Kronecker tensor**: $g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}{}_{\nu}$
- Raising both indices on $g_{\mu\nu}$ gives the inverse tensor $g^{\lambda\mu} g^{\kappa\nu} g_{\mu\nu} = g^{\lambda\kappa}$ and lowering both indices on $g^{\lambda\kappa}$ gives the metric tensor $g_{\mu\nu}$

Tensor algebra IV

- Contraction on a pair of indices that are either both contravariant or both covariant is not possible in general, but it is possible in the presence of the metric tensor $g_{\mu\nu}$
- **Metric contraction**: a combined operation of using the metric tensor to raise or lower one of the indices, as needed, followed by the usual operation of contraction
- **Scalar** (**inner** or **dot**) product of two vectors is:
$$A \cdot B = (A^{\mu}, B^{\nu}) = g_{\mu\nu} A^{\mu} B^{\nu} = A^{\mu} B_{\mu},$$
where the final result was obtained by lowering the contravariant index ν
- Scalar product of two vectors is a scalar and if it is equal to 0, then the two vectors are referred to as **orthogonal**
- The **square** of a vector is scalar product of the vector with itself
- **Inner product** of two tensors (a generalization of the scalar product of vectors) is obtained by taking the direct product of two tensors for the special case where one index is repeated, and taking the sum over this repeated index (contraction)
- Resulting tensor that has rank equal to the sum of the original ranks reduced by 2 for one contraction
- **The quotient theorem** (criterion for tensor character): if the result of taking the product (direct or inner) of a given set of elements with a tensor of any specified type and arbitrary components is known to be a tensor, then the given elements are the components of a tensor

Symmetric and antisymmetric tensors

- Tensor is **symmetric** in any of its indices if it is unchanged under exchange of those indices
  - Example 1: if $S_{\mu\nu\rho} = S_{\nu\mu\rho}$ then $S_{\mu\nu\rho}$ is symmetric in its first two indices
  - Example 2: if $S_{\mu\nu\rho} = S_{\mu\rho\nu} = S_{\rho\mu\nu} = S_{\nu\mu\rho} = S_{\nu\rho\mu} = S_{\rho\nu\mu}$ then $S_{\mu\nu\rho}$ is symmetric in all three of its indices
- Tensor is **antisymmetric** in any of its indices if it changes sign when those indices are exchanged
  - Example: if $A_{\mu\nu\rho} = -A_{\rho\nu\mu}$ then $A_{\mu\nu\rho}$ is antisymmetric in its first and third indices (or simply antisymmetric in μ and ρ)
- Tensor which is (anti-) symmetric in all of its indices, is referred to as **completely (anti-) symmetric**
- Examples:
  - Metric and the inverse metric tensors are symmetric
  - **Levi-Civita tensor** $\epsilon_{\mu\nu\rho\sigma}$ is completely antisymmetric:

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of 0123} \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of 0123} \\ 0 & \text{otherwise} \end{cases}$$

Exam questions

1. Tensors, their transformation rules and metric tensor
2. Tensor algebra: direct product, linear combination, contraction and trace, raising and lowering indices, scalar product of two vectors and inner product of tensors, symmetric and antisymmetric tensors

Literature

- Textbook: Weinberg, S., 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, Wiley-VCH

Exercise 1

- For a contravariant vector $V^\alpha = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and a covariant vector $X_\beta = [e, f, g, h]$ evaluate:

  a) their direct product: $T^\alpha{}_\beta = V^\alpha X_\beta$

  b) value of $T^3{}_0$ component of the direct product

Exercise 2

- A two index "object" $X^{\mu\nu}$ is defined by the "direct sum"of two vectors $X^{\mu\nu} = A^\mu + B^\nu$. Is $X^{\mu\nu}$ a tensor? Is there a transformation law to take X to a new coordinate system, i.e. to obtain $X^{\mu'\nu'}$ from $X^{\mu\nu}$?

Exercise 3

- Evaluate the following direct products of the tensors in the matrix form:

a) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}$

Exercise 4

- Calculate the components of covariant metric tensor $g_{\mu\nu}$ and contravariant (inverse) metric tensor $g^{\mu\nu}$ for:

a) two-dimensional flat Euclidean space (Euclidean plane) in polar coordinates, with metric: $ds^2 = dr^2 + r^2 d\theta^2$

b) two-sphere metric: $ds^2 = a^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$

Exercise 5

- For the two tensors $V^{\mu\nu}{}_{\varepsilon}$ and $W^{\gamma}{}_{\alpha\beta}$ evaluate their:

  a) direct product

  b) inner product for $\varepsilon = \gamma$

  c) inner product for $\varepsilon = \gamma$ and $\alpha = \nu$

Exercise 6

- Find the covariant vector A_μ associated to the contravariant vector $\mathrm{A}^{\mu} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$,

  as well as its square A^2 in Minkowski spacetime where: $\mathrm{g}_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 7

- Write the general forms of a symmetric tensor $S_{\mu\nu} = S_{\nu\mu}$ and an asymmetric tensor $A_{\mu\nu} = -A_{\nu\mu}$ as matrices of latin letters *a, b, c, …*

Solution 1

a) $T^{\alpha}{}_{\beta} = V^{\alpha} X_{\beta} = \begin{bmatrix} ae & af & ag & ah \\ be & bf & bg & bh \\ ce & cf & cg & ch \\ de & df & dg & dh \end{bmatrix}$

b) $T^{3}{}_{0} = V^{3} X_{0} = de$

Solution 2

If $X^{\mu\nu} = A^\mu + B^\nu$ and A^μ and B^ν are vectors, then this should hold:

$$X'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho} A^\rho + \frac{\partial x'^\nu}{\partial x^\sigma} B^\sigma$$

It is not possible to express this as $X'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} X^{\rho\sigma}$, and therefore there is no transformation law to take X to a new coordinate system and X cannot be a tensor

Solution 3

a) $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ a_{21}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1\times 0 & 1\times 5 & 2\times 0 & 2\times 5 \\ 1\times 6 & 1\times 7 & 2\times 6 & 2\times 7 \\ 3\times 0 & 3\times 5 & 4\times 0 & 4\times 5 \\ 3\times 6 & 3\times 7 & 4\times 6 & 4\times 7 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}$$

Solution 4

a)

$$g_{rr} = 1, \quad g_{r\theta} = 0, \quad g_{\theta r} = 0, \quad g_{\theta\theta} = r^2 \qquad \Rightarrow \quad g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

$$g^{rr} = 1, \quad g^{r\theta} = 0, \quad g^{\theta r} = 0, \quad g^{\theta\theta} = r^{-2} \qquad \Rightarrow \quad g^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}$$

b)

$$g_{\theta\theta} = a^2, \quad g_{\theta\phi} = 0, \quad g_{\phi\theta} = 0, \quad g_{\phi\phi} = a^2 \sin^2\theta \Rightarrow g_{\mu\nu} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{bmatrix}$$

$$g^{\theta\theta} = a^{-2}, \quad g^{\theta\phi} = 0, \quad g^{\phi\theta} = 0, \quad g^{\phi\phi} = a^{-2} \sin^{-2}\theta \Rightarrow g^{\mu\nu} = \begin{bmatrix} a^{-2} & 0 \\ 0 & a^{-2} \sin^{-2}\theta \end{bmatrix}$$

Solution 5

a) $V^{\mu\nu}{}_{\varepsilon} W^{\gamma}{}_{\alpha\beta} = U^{\mu\nu\gamma}{}_{\varepsilon\alpha\beta}$

b) $V^{\mu\nu}{}_{\varepsilon} W^{\varepsilon}{}_{\alpha\beta} = U^{\mu\nu}{}_{\alpha\beta}$

c) $V^{\mu\nu}{}_{\varepsilon} W^{\varepsilon}{}_{\nu\beta} = U^{\mu}{}_{\beta}$

Solution 6

$$A_\mu = g_{\mu\nu} A^\nu = \begin{bmatrix} -1, & -1, & 2, & 3 \end{bmatrix}$$

$$A^2 = -(1)^2 + (-1)^2 + 2^2 + 3^2 = 13$$

Solution 7

$$S_{\mu\nu} = S_{\nu\mu} = \begin{bmatrix} w & a & b & c \\ a & x & d & e \\ b & d & y & f \\ c & e & f & z \end{bmatrix}, \qquad A_{\mu\nu} = -A_{\nu\mu} = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$