MASS 2026 Course:
Gravitation and Cosmology

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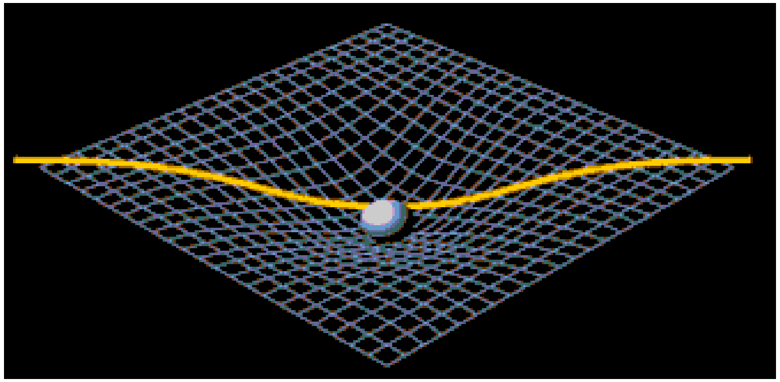
Lecture 04

- Locally inertial frame in GR
- Basic principles of GR:
  1. The principle of equivalence (strong and weak)
  2. The principle of general covariance (the general principle of relativity)
- Affine connection
- Geodesic equation
- Newtonian limit in GR
- Exercises

Basics of GR: locally inertial frame

- GR is formulated by Albert Einstein in 1915 and is currently accepted as the standard theory of gravity
- GR considers curved spacetimes as 4-dimensional Riemannian manifolds and studies accelerated relative motions and gravity
- The presence of matter curves spacetime, and this curvature affects the paths of free particles and light



- There is no global inertial frame in GR, however it is possible to choose a locally inertial frame at every point in spacetime
- **Locally inertial frame** is a reference frame within a sufficiently small region around the given point, in which the laws of nature take the same form as in inertial reference frame in SR, i.e. in absence of gravitation
- In a general coordinate system x^μ, metric $g_{\mu\nu}$ is defined by: $g_{\mu\nu} \equiv \eta_{\alpha\beta} \dfrac{\partial \xi^\alpha}{\partial x^\mu} \dfrac{\partial \xi^\beta}{\partial x^\nu}$, where ξ^α is a locally inertial coordinate system
- Metric $g_{\mu\nu}$ can be locally transformed to the Minkowski metric $\eta_{\mu\nu}$: for every point P in the spacetime, there is a coordinate transformation that makes $g_{\mu\nu} = \eta_{\mu\nu}$ at P
- GR postulates that, in the presence of matter, the global Lorentz covariance of SR becomes a local Lorentz covariance in an infinitesimal region of spacetime at every point, so that SR controls physics only locally
- In GR: *gravitation = a field of locally inertial frames*

Basic principles of GR

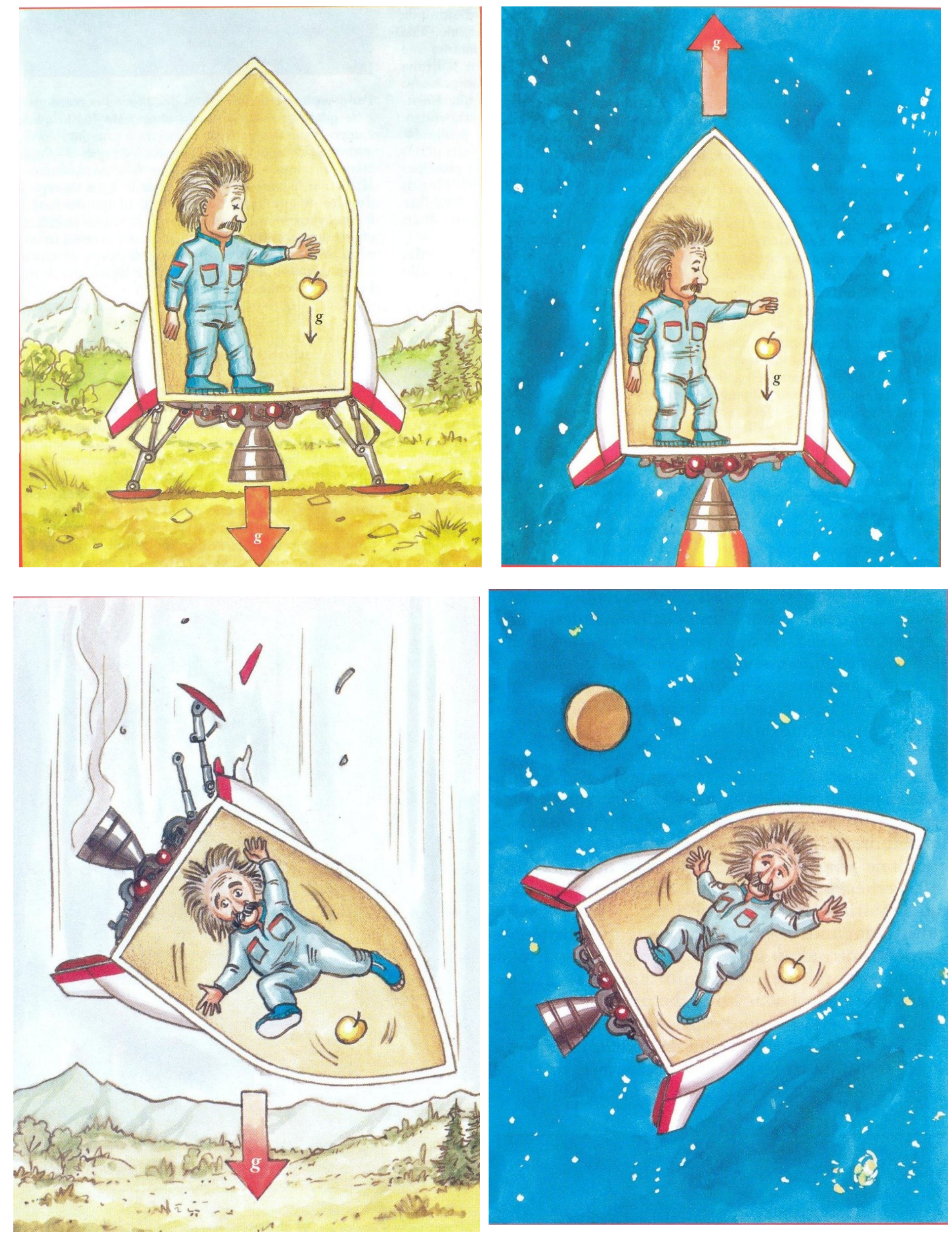
- GR is based on the following two **principles**:

1. **Principle of equivalence** or **strong equivalence principle (EP)**: at every space-time point in an arbitrary gravitational field it is possible to choose a locally inertial reference frame such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in inertial reference frame in the absence of gravitation (i.e. as in SR)
   - **Weak equivalence principle (WEP)** is another formulation which states the same, but refers to the laws of motion of freely falling bodies, instead of all physical laws
2. **Principle of general covariance (PGC)** or **general principle of relativity:** a physical equation holds in a general gravitational field, if two conditions are met:
   i. The equation holds in the absence of gravitation, i.e. it agrees with the laws of SR when the metric tensor $g_{\mu\nu}$ equals the Minkowski tensor $\eta_{\mu\nu}$
   ii. The equation is generally covariant, i.e. it preserves its form under a general coordinate transformation $x \to x'$

- According to the 2nd condition of PGC, the equation will be true in all coordinate systems if it is true in any one coordinate system
- According to the 1st condition of PGC, the equation holds in the locally inertial systems in which the effects of gravitation are absent, and hence in all other coordinate systems

Equivalence Principle (EP)

- EP in Newtonian gravity rests on the equality of gravitational and inertial mass: $m_g = m_i$
- EP in GR represents its physical basis and rests on these Einstein's observations:
  - Gravitational force as experienced locally while standing on a massive body (such as the Earth) is the same as the pseudo-force experienced by an observer in a non-inertial (accelerated) frame of reference
  - An observer in free fall in a gravitational field feels the same laws of physics as an observer which is not in a gravitational field (e.g. like astronauts in space)
- Locally, acceleration is equivalent to a gravitational field, and its effects are indistinguishable from the effects of gravity
- Bodies far away from any gravitational field move along the straight lines, and those in a gravitational field move along the shortest spacetime lines (geodesics), due to the curvature of spacetime caused by the presence of gravitational masses



Principle of General Covariance

- The main significance of PGC lies in its statement about the effects of gravitation, that a physical equation will be true in a gravitational field if it is true in the absence of gravitation
- Since PE assures that a coordinate system in which the effects of gravitation are absent can be constructed only on a scale that is small compared with the spacetime distances typical of the gravitational field, PGC can only be applied on such small scales
- According to PGC, any equation can be made generally covariant by writing it in any one coordinate system, and then transforming it to other arbitrary coordinate systems
- To ensure their general covariance, equations in GR are constructed using tensors
- Because PGC is applied on a small scale compared with the scale of the gravitational field, only metric tensor $g_{\mu\nu}$ and its first derivatives are expected to enter the generally covariant equations

Affine connection

- Riemannian geometry is the study of manifolds with metrics and their associated connections (additional structures which are characterized by the curvature)
- The existence of a metric implies a certain connection, whose curvature may be thought of as that of the metric
- GR is based on the connection derived from $g_{\mu\nu}$ which is known as **affine connection**, **Christoffel connection**, **Levi-Civita connection** or **Riemannian connection**, and its coefficients are called **Christoffel symbols** of the second kind
- **Affine connection** defined in terms of the metric tensor $g_{\mu\nu}$ is:

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right) = \frac{1}{2} g^{\nu\sigma} \left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu} \right)$$

- The connection coefficients are symmetric in the two lower indices, but are not the components of a tensor
- In GR, affine connection $\Gamma^{\lambda}_{\mu\nu}$ represents generalization of the Newtonian gravitational force, and metric tensor $g_{\mu\nu}$ is generalization of the Newtonian gravitational potential (derivatives of $g_{\mu\nu}$ determine the force $\Gamma^{\lambda}_{\mu\nu}$)

Geodesic equation

- In GR, a geodesic generalizes the notion of a "straight line" to curved spacetime
- A freely moving or falling particle (i.e. a particle free from all external, non-gravitational forces) always moves along a geodesic
- Geodesic equation can be derived directly from the Equivalence Principle or via the action principle, and it is given by: $$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0,$$

  where *s* is a scalar parameter of motion (e.g. the proper time), and $\Gamma^\mu_{\nu\lambda}$ are affine connection coefficients (Christoffel symbols)
- In flat space $\Gamma^\mu_{\nu\lambda} = 0$ and geodesics reduce to straight lines defined by: $\frac{d^2x^\mu}{ds^2} = 0$
- It is not necessary to know what *s* is in order to find the motion of a particle, for geodesic equation when solved gives $x^\mu(s)$, and *s* can be eliminated to give $x(t)$
- Explicit expression for a geodesic can be found only when $g_{\mu\nu}$ is known
- The left-hand-side of geodesic equation is the acceleration of a particle, and thus geodesic equation is analogous to the Newton's laws of motion
- For a massless particle such as photon, the following initial conditions are imposed:

$$0 = -g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

Newtonian limit in GR

- Newtonian limit in GR is obtained for a particle moving slowly in a weak stationary gravitational field
- For a sufficiently **slow particle**, derivatives $d\mathbf{x}/ds$ may be neglected with respect to derivatives cdt/ds, and then the geodesic equation is: $\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{00}\left(\frac{c\,dt}{ds}\right)^2 = 0$
- In a **stationary field** all time derivatives of $g_{\mu\nu}$ vanish, so: $\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu}$
- Metric in the **weak field** is equal to Minkowski metric plus a small perturbation:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad |h_{\alpha\beta}| \ll 1,$$

so to the first order in $h_{\alpha\beta}$: $\Gamma^\alpha_{00} = -\frac{1}{2}\eta^{\alpha\beta}\frac{\partial h_{00}}{\partial x^\beta}$ $\Rightarrow$ the equations of motion are:

$$\frac{d^2\mathbf{x}}{ds^2} = \frac{1}{2}\left(\frac{c\,dt}{ds}\right)^2 \nabla h_{00} \quad \Bigg| \cdot \left(\frac{ds}{c\,dt}\right)^2 \quad \Rightarrow \quad \frac{d^2\mathbf{x}}{c^2dt^2} = \frac{1}{2}\nabla h_{00}$$

- The corresponding Newtonian result is: $\frac{d^2\mathbf{x}}{dt^2} = -\nabla\phi \quad \Rightarrow \quad h_{00} = -\frac{2\phi}{c^2}$,

where ϕ is the Newton's gravitational potential: $\phi = -\frac{GM}{r}$

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad \Rightarrow \quad \boxed{g_{00} = -\left(1 + \frac{2\phi}{c^2}\right)}$$

Exam question

1. Basic principles of GR: the principle of equivalence and the principle of general covariance
2. Affine connection and geodesic equation

Literature

- Weinberg, S., 1972, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley-VCH
- Sean M. Carroll, 1997. *Lecture Notes on General Relativity*, arXiv, gr-qc/9712019

Exercise 1

- Calculate all affine connection coefficients (Christoffel symbols) $\Gamma^{\alpha}_{\beta\gamma}$ for the two-dimensional flat Euclidean space (Euclidean plane) in polar coordinates, with metric: $ds^2 = dr^2 + r^2 d\theta^2$.
  Do all affine connection coefficients of a flat space vanish in curvilinear coordinate systems?

Exercise 2

- Calculate all nonzero affine connection coefficients (Christoffel symbols) $\Gamma^{\alpha}_{\beta\gamma}$ for the two-sphere metric: $ds^2 = a^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$

Exercise 3

- Consider again the metric $ds^2 = dr^2 + r^2 d\theta^2$ and write the 2 equations that result from the geodesic equation

Exercise 4

- For the 2-dimensional metric $ds^2 = (dx^2 - dt^2)/t^2$, find all nonvanishing affine connection coefficients $\Gamma^{\mu}_{\nu\lambda}$ and derive the geodesic curves

Solution 1

- Nonzero components of metric tensor and inverse metric tensor:

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2 \quad \wedge \quad g^{rr} = 1, \quad g^{\theta\theta} = r^{-2} \quad \Rightarrow$$

- The only nonzero derivative of metric tensor is: $g_{\theta\theta,r} = 2r$

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu} \right) \quad \Rightarrow$$

$$\begin{aligned}
\Gamma^{r}_{rr} &= \frac{1}{2} g^{r\rho} (g_{r\rho,r} + g_{\rho r,r} - g_{rr,\rho}) \\
&= \frac{1}{2} g^{rr} (g_{rr,r} + g_{rr,r} - g_{rr,r}) + \frac{1}{2} g^{r\theta} (g_{r\theta,r} + g_{\theta r,r} - g_{rr,\theta}) \\
&= \frac{1}{2} \cdot 1 \cdot (0 + 0 - 0) + \frac{1}{2} \cdot 0 \cdot (0 + 0 - 0) = 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^{r}_{\theta\theta} &= \frac{1}{2} g^{r\rho} (g_{\theta\rho,\theta} + g_{\rho\theta,\theta} - g_{\theta\theta,\rho}) = \frac{1}{2} g^{rr} (g_{\theta r,\theta} + g_{r\theta,\theta} - g_{\theta\theta,r}) \\
&= \frac{1}{2} \cdot 1 \cdot (0 + 0 - 2r) = -r
\end{aligned}$$

- In the similar way: $\Gamma^{r}_{\theta r} = \Gamma^{r}_{r\theta} = 0, \quad \Gamma^{\theta}_{rr} = 0, \quad \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \quad \Gamma^{\theta}_{\theta\theta} = 0$
- As it can be seen from this example, not all affine connection coefficients of a flat space vanish in curvilinear coordinate systems

Solution 2

- Nonzero components of metric tensor and inverse metric tensor:
  $g_{\theta\theta} = a^2, \quad g_{\phi\phi} = a^2 \sin^2\theta \quad \wedge \quad g^{\theta\theta} = a^{-2}, \quad g^{\phi\phi} = a^{-2}\sin^{-2}\theta$
- Nonzero Christoffel symbols: $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$

Solution 3

According to the solution of the Exercise 1, the nonzero affine connection coefficients for this metric are: $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \dfrac{1}{r}$

Therefore, the geodesic equation is: $\dfrac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\dfrac{dx^\nu}{ds}\dfrac{dx^\lambda}{ds} = 0 \Rightarrow$

$$\frac{d^2r}{ds^2} + \Gamma^r_{\theta\theta}\frac{d\theta}{ds}\frac{d\theta}{ds} = 0 \ \wedge \ \frac{d^2\theta}{ds^2} + 2\Gamma^\theta_{r\theta}\frac{dr}{ds}\frac{d\theta}{ds} = 0 \ \Rightarrow$$

$$\frac{d^2r}{ds^2} = r\left(\frac{d\theta}{ds}\right)^2 \ \wedge \ \frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} = 0$$

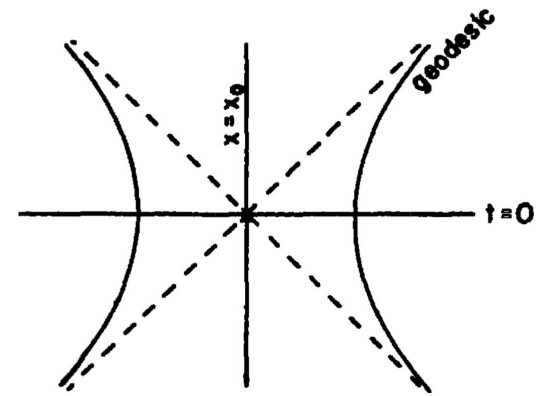
Solution 4

- Nonzero components of metric tensor and inverse metric tensor:

$$g_{tt} = -t^{-2}, \quad g_{xx} = t^{-2} \quad \wedge \quad g^{tt} = -t^{2}, \quad g^{xx} = t^{2} \quad \Rightarrow$$

- Nonzero derivatives of metric tensor: $g_{tt,t} = 2t^{-3}, \quad g_{xx,t} = -2t^{-3}$
- Nonzero Christoffel symbols: $\Gamma^{t}_{tt} = \Gamma^{t}_{xx} = \Gamma^{x}_{tx} = \Gamma^{x}_{xt} = -\dfrac{1}{t}$
- Geodesics: $\dfrac{d^2x^\mu}{ds^2} + \Gamma^{\mu}_{\nu\lambda}\dfrac{dx^\nu}{ds}\dfrac{dx^\lambda}{ds} = 0 \quad \overset{x^\mu = t}{\Longrightarrow} \quad \dfrac{d^2t}{ds^2} = \Gamma^{t}_{tt}\left(\dfrac{dt}{ds}\right)^2 + \Gamma^{t}_{xx}\left(\dfrac{dx}{ds}\right)^2 = 0$

$$\frac{d^2x^\mu}{ds^2} + \Gamma^{\mu}_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0 \quad \overset{x^\mu = x}{\Longrightarrow} \quad \frac{d^2x}{ds^2} = 2\Gamma^{x}_{tx}\frac{dt}{ds}\frac{dx}{ds} = 0$$



$$\frac{d^2t}{ds^2} = \frac{1}{t}\left[\left(\frac{dt}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2\right] \quad (1) \qquad \frac{d^2x}{ds^2} = \frac{2}{t}\frac{dt}{ds}\frac{dx}{ds} \quad (2)$$

$$(2) \Rightarrow \frac{d}{ds}\left(\frac{dx}{ds}\right) = \frac{d}{ds}\left(\ln t^2\right)\frac{dx}{ds} \;\Leftrightarrow\; \frac{\dfrac{d}{ds}\left(\dfrac{dx}{ds}\right)}{\dfrac{dx}{ds}} = \frac{d}{ds}\left(\ln t^2\right) \;\Leftrightarrow$$

$$\frac{d}{ds}\left(\ln\frac{dx}{ds}\right) = \frac{d}{ds}\left(\ln t^2\right) \Rightarrow \frac{dx}{ds} = c\,t^2 \quad (3) \quad \overset{\text{metric}}{\Longrightarrow} \quad \left(\frac{dt}{ds}\right)^2 = c^2t^4 - t^2 \Rightarrow$$

$$\frac{dt}{ds} = \pm t\sqrt{c^2t^2 - 1} \quad \overset{(3)}{\Longrightarrow} \quad \frac{dx}{dt} = \pm\frac{c\,t}{\sqrt{c^2t^2-1}} \Rightarrow x - x_0 = \pm\sqrt{t^2 - c^{-2}} \Rightarrow$$

- Geodesics are hyperbolas asymptotic to the light cones: $\dfrac{t^2}{(1/c)^2} - \dfrac{(x-x_0)^2}{(1/c)^2} = 1$