

**MASS 2026 Course:**  
**Gravitation and Cosmology**

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# Lecture 06

- Lagrangian and action
- Hamilton's principle of least (stationary) action
- Einstein-Hilbert action
- Matter action
- Calculus of variations
  - Functional and its variation
  - Extremals
  - Euler-Lagrange equation
  - Variational (or functional) derivative
- Derivation of the EFE using the principle of stationary action:
  - Variation of the action with respect to the inverse metric
- Exercises

# Lagrangian and action

- Three mathematically equivalent formalisms for classical mechanics: Newtonian with vectors (forces and accelerations), Lagrangian with scalars (kinetic and potential energy) and Hamiltonian with position and momentum in phase space
- Lagrangian formulation is most suitable for deriving the equations of motion (EoM) for complex physical systems and in field theory
- EoM are derived using the **Hamilton's Principle of Least Action**, which states that a physical system follows a path that minimizes (or makes stationary) the action  $S$
- **Action**  $S$  is a functional (a function of a path) defined as the time integral of

Lagrangian  $L$  along the path: 
$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt$$

- **Lagrangian**  $L(q_i, \dot{q}_i, t)$  is a function defined as the difference between the kinetic energy  $T$  and potential energy  $V$  of a system ( $L = T - V$ ), expressed in terms of generalized coordinates  $q_i$  and velocities  $\dot{q}_i$
- Action  $S$  is a scalar quantity with dimensions of [energy]  $\times$  [time], i.e.  $[S] = \text{J} \cdot \text{s}$ . that characterizes the trajectory or evolution of a physical system over time
- EoM are derived by minimizing the action  $S$  using the calculus of variations ( $\delta S = 0$ ), leading to the **Euler-Lagrange equation**: 
$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

# Principle of least (stationary) action

- **Hamilton's principle of least (stationary) action** is a fundamental principle stating that the path a physical system takes between two states over a given time interval is the one that makes the action  $S$  stationary (usually a minimum, but sometimes a maximum or saddle point)
- It is a variational principle (expressed as  $\delta S = 0$ ) that, when applied to the action of a physical system, allows for the derivation of the equations of motion (Euler-Lagrange equations) for that system
- It is a foundational principle in classical mechanics, electromagnetism, general relativity, quantum mechanics, particle physics, ...
- Geodesics can be derived using the principle of stationary action by finding the path  $x^\mu(\lambda)$  that extremizes the spacetime interval (or proper time) between two points:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \Rightarrow S = \int ds = \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

- Lagrangian in this case is given by:  $L(x^\mu, \dot{x}^\mu) = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$  where  $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ , and then the geodesic  $x^\mu(\lambda)$  can be obtained from the Euler-Lagrange equation:

$$\frac{\partial L}{\partial x^\sigma} - \frac{d}{d\lambda} \left( \frac{\partial L}{\partial \dot{x}^\sigma} \right) = 0$$

# Einstein-Hilbert action

- In contrast to mechanics which describes the motion of discrete particles or bodies, field theory studies continuous fields that fill space
- In a field theory, the Lagrangian is integral over space of a **Lagrangian density**  $\mathcal{L}$ , which is a function of the fields  $\phi^i(x^\mu)$ , their derivatives and spacetime coordinates:  
$$L = \int d^3x \mathcal{L}(\phi^i, \phi^i_{,\mu}, x^\mu) \Rightarrow \text{the action is: } S = \int dt L = \int d^4x \mathcal{L}(\phi^i, \phi^i_{,\mu}, x^\mu)$$
- GR is a classical field theory, generalized to curved spacetime by replacing partial derivatives  $\phi^i_{,\mu}$  with covariant ones  $\phi^i_{;\mu}$ :  $S = \int \mathcal{L}(\phi^i, \phi^i_{;\mu}, x^\mu) d^4x$
- Field equations of GR were first derived by Hilbert using the action principle, by taking into account that metric  $g_{\mu\nu}$  is the field variable and invariant volume element is the scalar  $\sqrt{-g} d^4x$ , where  $g = \det g_{\mu\nu}$
- Hilbert figured that the simplest possible choice for Lagrangian density is the one which is proportional to Ricci scalar  $R$ :  $\mathcal{L}_H = \frac{R}{2\kappa}$ , where  $\kappa = \frac{8\pi G}{c^4}$  is the **Einstein gravitational constant**
- **Einstein-Hilbert action** is then:  $S_H = \int \sqrt{-g} d^4x \mathcal{L}_H = \frac{1}{2\kappa} \int R \sqrt{-g} d^4x$
- $S_H$  represents the gravitational part of the full action  $S$  in GR, and its variation with respect to the metric  $g_{\mu\nu}$  leads to the Einstein field equations (EFE) in vacuum:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

# Matter action

- Non-vacuum EFE are derived from the total action  $S$  which combines the Einstein-Hilbert action  $S_H$  for gravity and a **matter action**  $S_M$ :  $S = S_H + S_M$
- $S_M$  in GR describe how non-gravitational fields (matter/energy) behave within curved spacetime and generate gravitational fields through the energy-momentum tensor  $T_{\mu\nu}$
- $S_M$  are defined as:  $S_M = \int \mathcal{L}_M \sqrt{-g} d^4x$ , where  $\mathcal{L}_M$  is the matter Lagrangian density
- In GR,  $T_{\mu\nu}$  is defined as **functional derivative** of  $S_M$  with respect to the inverse metric  $g^{\mu\nu}$  (the Hilbert stress-energy tensor):  $T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_M)}{\delta g^{\mu\nu}}$
- For the perfect fluid:  $\mathcal{L}_M = -p$  (negative pressure of the fluid), and the resulting energy-momentum tensor is:  $T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U_\mu U_\nu + p g_{\mu\nu}$
- In the most general case the cosmological constant  $\Lambda$  is also included in  $\mathcal{L}$  :  
$$\mathcal{L} = \frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_M,$$
and the total action is then given by:  $S = \int \left[ \frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_M \right] \sqrt{-g} d^4x$
- Variation of the total action with respect to the metric leads to the non-vacuum EFE in the general form:  $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$

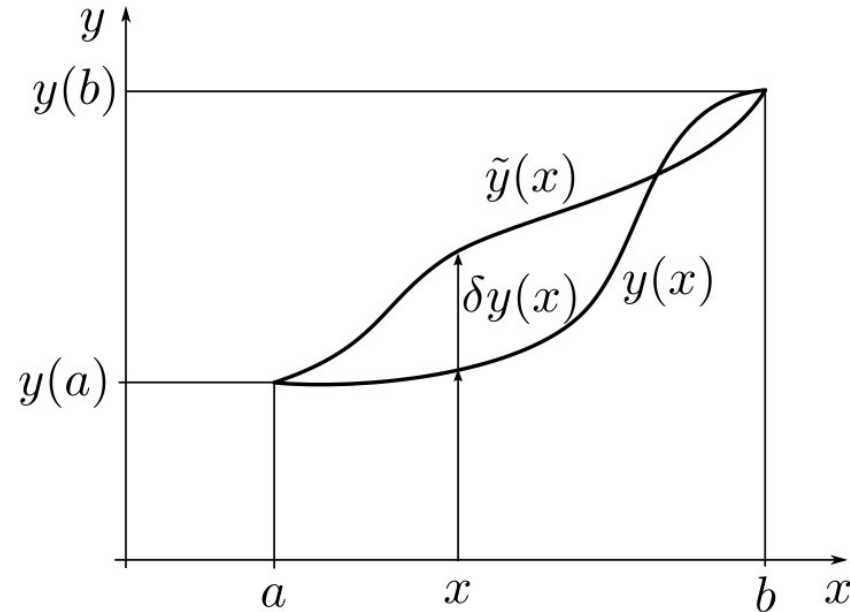
# Calculus of variations

- **Calculus of variations** is used for finding the maxima and minima of functionals

- **Functional**  $J[y]$  is a correspondence  $J: y(x) \rightarrow R$  which assigns a real number to each function (or curve)  $y = y(x)$  belonging to some class

- Functional is a kind of function, where the independent variable is itself a function

- Example: arc length of a curve  $y = y(x)$  in Euclidean plane:  $J[y] = \int_a^b \sqrt{1 + y'^2} dx$



- More general functionals of the following form are of particular importance:

$$J[y] = \int_a^b F[x, y(x), y'(x)] dx, \quad y(a) = A, \quad y(b) = B,$$

where  $F(x, y, z)$  is a continuous function of three variables, and  $y(x)$  is a continuously differentiable function defined on the interval  $[a, b]$

- **Increment**  $\Delta J$  of the functional  $J[y]$  that corresponds to the increment (perturbation)  $h = h(x)$  of the "independent variable"  $y = y(x)$ :

$$\Delta J = J[y + h] - J[y] = \int_a^b [F(x, y + h, y' + h') - F(x, y, y')] dx$$

# Euler-Lagrange differential equation

- **Variation  $\delta J$  of functional  $J[y]$**  is the first, linear term in the Taylor expansion of  $\Delta J$ :

$$\delta J = \int_a^b [F_y(x, y, y') h + F_{y'}(x, y, y') h'] dx = \int_a^b \left( \frac{\partial F}{\partial y} h + \frac{\partial F}{\partial y'} h' \right) dx$$

- **Extremal** is the function  $y = y(x)$  for which functional  $J[y]$  has an extremum
- A necessary condition for  $J[y]$  to have an extremum for  $y = y(x)$  is:

$$\delta J = 0 \quad \Rightarrow \quad \boxed{\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0} \quad \text{(Euler-Lagrange differential equation)}$$

- Extremals are obtained by solving the Euler-Lagrange differential equation
- Solution of this second-order differential equation will depend on two arbitrary constants, which are determined from the boundary conditions:  $y(a) = A$  and  $y(b) = B$
- Special cases where Euler-Lagrange equation can be reduced to a first-order differential equation, or where its solution can be obtained by evaluating integrals:

1.  $F$  does not depend on  $y$ :  $F = F(x, y') \Rightarrow \frac{\partial F}{\partial y'} = C \Rightarrow y' = f(x, C)$

2.  $F$  does not depend on  $x$ :  $F = F(y, y') \Rightarrow$  The first integral is:  $F - y' \frac{\partial F}{\partial y'} = C$

3.  $F$  does not depend on  $y'$ :  $F = F(x, y) \Rightarrow \frac{\partial F}{\partial y} = 0$  (is not a differential equation)

# Variational derivative

- **Variational (functional) derivative** relates a change in a functional to a change in a function on which the functional depends
- In the case of functionals of the type:  $J[y] = \int_a^b F[x, f(x), f'(x)] dx$ , variational derivative is the left-hand side of Euler-Lagrange equation:  $\frac{\delta J}{\delta f} = \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'}$
- Functional  $J[y]$  has an extremum if its variational derivative vanishes at every point, like in the case of a function which has an extremum if all of its partial derivatives vanish
- The analogs of all the familiar rules obeyed by ordinary derivatives (e.g. the formulas for differentiating sums and products of functions, composite functions, etc.) are also valid for variational derivatives:

– Linearity:  $\frac{\delta(\lambda F + \mu G)[\rho]}{\delta \rho(x)} = \lambda \frac{\delta F[\rho]}{\delta \rho(x)} + \mu \frac{\delta G[\rho]}{\delta \rho(x)}$ , where  $\lambda$  and  $\mu$  are constants

– Product rule:  $\frac{\delta(FG)[\rho]}{\delta \rho(x)} = \frac{\delta F[\rho]}{\delta \rho(x)} G[\rho] + F[\rho] \frac{\delta G[\rho]}{\delta \rho(x)}$

– Chain rules:  $\frac{\delta F[G[\rho]]}{\delta \rho(y)} = \int dx \frac{\delta F[G]}{\delta G(x)} \Big|_{G=G[\rho]} \cdot \frac{\delta G[\rho](x)}{\delta \rho(y)}$

$$\frac{\delta F[g(\rho)]}{\delta \rho(y)} = \frac{\delta F[g(\rho)]}{\delta g[\rho(y)]} \frac{dg(\rho)}{d\rho(y)}$$

- Functional derivative with respect to the metric is used for variation of the Hilbert action in order to obtain the Einstein field equations

# Derivation of the EFE

- According to the principle of least action, the variation of the action with respect to the

inverse metric is zero:  $\delta S = \delta \left\{ \int \left[ \frac{1}{2\kappa} R + \mathcal{L}_M \right] \sqrt{-g} d^4x \right\} = 0 \quad \Leftrightarrow$

$$\int \left[ \frac{1}{2\kappa} \frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} d^4x = 0 \quad \Leftrightarrow$$

$$\int \left[ \frac{1}{2\kappa} \left( \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \sqrt{-g} d^4x = 0 \quad \Rightarrow$$

$$\frac{1}{2\kappa} \left( \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = 0 \quad \Rightarrow$$

$$\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \frac{-2\kappa}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \Leftrightarrow \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \kappa T_{\mu\nu} \quad (1)$$

- Variation of the Ricci scalar:  $\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \quad (2)$

- Variation of the metric determinant:  $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}(g_{\mu\nu}\delta g^{\mu\nu}) \quad (3)$

- **Einstein field equations:**  $(1) \wedge (2) \wedge (3) \quad \Rightarrow \quad \boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}}$

- In the case with cosmological constant  $\Lambda$ :  $\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}}$

- EFE relate the geometry of spacetime to the distribution of matter within it

- Wheeler: "Matter tells spacetime how to curve. Spacetime tells matter how to move."

# Exam questions

1. Lagrangian, action, Hamilton's principle of least (stationary) action, Einstein-Hilbert action and matter action
2. Calculus of variations, Euler-Lagrange equation, variational (functional) derivative and derivation of the Einstein field equations using the principle of stationary action

## Literature

- Sean Carroll, 2004, *Spacetime and Geometry: An Introduction to General Relativity*, Addison Wesley
- I. M. Gelfand and S. V. Fomin, 1963, *Calculus of Variations*, Revised English Edition, Prentice-Hall, Inc

# Exercise 1

- Among all the curves  $y = f(x)$  in the Euclidean plane, find the one which has the shortest arc length  $A[y] = \int_{x_1}^{x_2} \sqrt{1 + [f'(x)]^2} dx$  between two given points  $(x_1, y_1)$  and  $(x_2, y_2)$ .

# Exercise 2

- Find the extremal of the following functional:

$$J[y] = \int_1^2 \frac{\sqrt{1 + y'^2}}{x} dx, \quad y(1) = 0, \quad y(2) = 1$$

# Exercise 3

- For the 2-dimensional metric  $ds^2 = (dx^2 - dt^2)/t^2$ , find all timelike geodesic curves using the principle of least action.

# Solution 1

- In order to find the extremal  $f(x)$  that minimizes the functional  $A[y]$  we have to solve the Euler-Lagrange equation:  $\frac{\partial L}{\partial f} - \frac{d}{dx} \frac{\partial L}{\partial f'} = 0$ , where:  $L = \sqrt{1 + [f'(x)]^2}$
- Since  $f$  does not appear explicitly in  $L$ , the first term in the Euler-Lagrange equation vanishes and thus:  $\frac{d}{dx} \frac{\partial L}{\partial f'} = 0 \Rightarrow \frac{d}{dx} \frac{f'(x)}{\sqrt{1 + [f'(x)]^2}} = 0 \Rightarrow$   
 $\frac{f'(x)}{\sqrt{1 + [f'(x)]^2}} = c \Rightarrow f'(x) = \frac{c}{\sqrt{1 - c^2}} = A \Rightarrow f(x) = Ax + B$
- Constants A and B are obtained from the boundary conditions:  
$$A = \frac{y_2 - y_1}{x_2 - x_1} \wedge B = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$
- The extremal  $f(x)$  that minimizes the functional  $A[y]$  is a straight line.

# Solution 2

$L = \frac{\sqrt{1 + y'^2}}{x}$  does not contain  $y$ , so Euler-Lagrange equation has the form:

$$\frac{d}{dx} \frac{\partial L}{\partial y'} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial y'} = C \quad \Rightarrow \quad \frac{y'}{x\sqrt{1 + y'^2}} = C \quad \Leftrightarrow$$

$$y'^2 (1 - C^2 x^2) = C^2 x^2 \quad \Leftrightarrow \quad y' = \frac{Cx}{\sqrt{1 - C^2 x^2}} \quad \Rightarrow \quad y = \frac{\sqrt{1 - C^2 x^2}}{C} + C_1$$

$$(y - C_1)^2 + x^2 = \frac{1}{C^2}$$

- From the boundary conditions, we find that:  $C = \frac{1}{\sqrt{5}}$ ,  $C_1 = 2$ , and the final solution is:  $(y - 2)^2 + x^2 = 5$

# Solution 3

- Let a geodesic be  $x(t)$ . Then the action is:  $S = \int ds = \int \sqrt{\dot{x}^2 - 1} \frac{dt}{t}$  and  $\delta S = \delta \int \sqrt{\dot{x}^2 - 1} \frac{dt}{t} = 0$
- In this case Euler-Lagrange equation is:  $\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$  and  $L = \frac{\sqrt{\dot{x}^2 - 1}}{t} \Rightarrow$   
 $\frac{d}{dt} \left( \frac{\dot{x}}{t\sqrt{\dot{x}^2 - 1}} \right) = 0 \Leftrightarrow \frac{\dot{x}}{t\sqrt{\dot{x}^2 - 1}} = c \Rightarrow \dot{x} = \pm \frac{ct}{\sqrt{c^2 t^2 - 1}} \Rightarrow$   
 $x - x_0 = \pm \sqrt{t^2 - c^{-2}} \Rightarrow t^2 - (x - x_0)^2 = \frac{1}{c^2} \Rightarrow \boxed{\frac{t^2}{(1/c)^2} - \frac{(x - x_0)^2}{(1/c)^2} = 1}$
- Geodesics are hyperbolas asymptotic to the light cones:

