MASS 2026 Course:

Gravitation and Cosmology

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Lecture 07

- Vacuum solutions to the field equations of GR
  - Schwarzschild metric
  - Kerr metric
  - Reissner-Nordström metric
  - Kerr-Newman metric
- Classic Solar System tests of GR:
  - Perihelion precession of Mercury's orbit
  - Deflection of light by the Sun
  - Gravitational redshift of light
  - Shapiro time delay
- Precession of orbiting gyroscopes (Lense-Thirring effect)

Schwarzschild metric

- Einstein field equations (EFE) are a system of 10 partial differential equations in which the metric tensor $g_{\mu\nu}$ can be solved for
- **Spacetime metrics** $g_{\mu\nu}$ are **solutions to EFE**, which can be exact or non-exact
- **Vacuum solutions** to EFE are those for which $T_{\mu\nu} = 0,$ and they describe region of spacetime where no matter, energy or non-gravitational fields are present
- **Schwarzschild metric**, found by Karl Schwarzschild in 1916, is an exact vacuum solution to EFE that describes the gravitational field outside a **spherically symmetric**, **non-rotating and uncharged** mass M

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) c^2 dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

  where $\boxed{R_S = \frac{2GM}{c^2}}$ is the **Schwarzschild radius**, and $d\Omega^2 = d\theta^2 + \sin^2\theta\ d\phi^2$ is the metric on a unit two-sphere
- Any object whose radius is smaller than its R_S is called a **black hole**
- In the case of a non-rotating black hole, the surface at R_S acts as an **event horizon** (boundary beyond which events cannot affect an observer)
- R_S of Sun is ~3 km, R_S of Earth is ~9 mm and R_S of Moon is ~0.1 mm
- Since $g_{00} = -\left(1 - \frac{2GM}{c^2 r}\right)$, Schwarzschild metric should reduce to the weak field case when $r \gg R_S$

Kerr metric

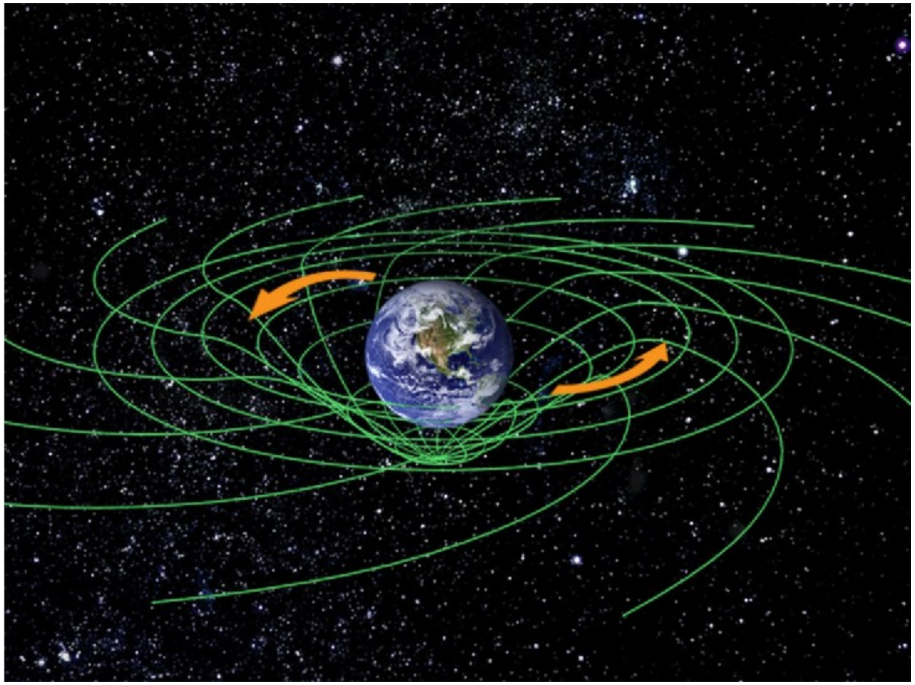
- **Kerr metric** is a generalization of the Schwarzschild metric to a rotating mass
- It is an exact vacuum solution to EFE, found by Roy Kerr in 1963, that describes the gravitational field around a **rotating**, **uncharged**, and **axially symmetric** mass *M*

$$ds^2 = -\left(1 - \frac{R_S r}{\Sigma}\right) c^2 dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{R_S r a^2}{\Sigma} \sin^2\theta\right) \sin^2\theta \, d\phi^2$$

$$- \frac{2 R_S r a \sin^2\theta}{\Sigma} c dt \, d\phi, \text{ where } \Sigma = r^2 + a^2 \cos^2\theta, \quad \Delta = r^2 - R_S r + a^2$$

- $a = \frac{J}{Mc}$ is the **spin** (angular momentum *J* normalized to mass *M)*
- **Event horizon** of a rotating black hole (in terms of R_S): $R_H^{\pm} = \frac{R_S \pm \sqrt{R_S^2 - 4a^2}}{2}$
- Rotating black holes have two horizons:
  1. **outer horizon** R_H^+ is the boundary of no return
  2. **inner (Cauchy) horizon** R_H^- is a deeper, highly unstable boundary beyond which spacetime becomes unpredictable
- For a=0 $\Rightarrow$ $R_H^+ = R_S$ and $R_H^- = 0$ (as in Schwarzschild case)
- **Frame-dragging**: in Kerr metric, the reference frame is pulled by the rotating central mass to co-rotate with it, having angular speed:

$$\Omega = \frac{R_S r a c}{\Sigma\left(r^2 + a^2\right) + R_S r a^2 \sin^2\theta}$$



Reissner-Nordström metric

- **Reissner–Nordström metric** is an exact solution to EFE, that describes the gravitational field around a charged, non-rotating and spherically symmetric mass M

$$ds^2 = \left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right) c^2\, dt^2 - \left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2,$$

where R_Q is a characteristic length scale given by $R_Q^2 = \frac{Q^2 G}{4\pi\varepsilon_0 c^4}$, that corresponds to the charge Q, and ε_0 is the vacuum permittivity (electric constant)

- Two concentric event horizons: $R_H^{\pm} = \frac{1}{2}\left(R_S \pm \sqrt{r_S^2 - 4R_Q^2}\right)$

Kerr-Newman metric

- **Kerr-Newman metric** is the most general asymptotically flat exact solution to EFE that describes the gravitational field around a charged and rotating mass M

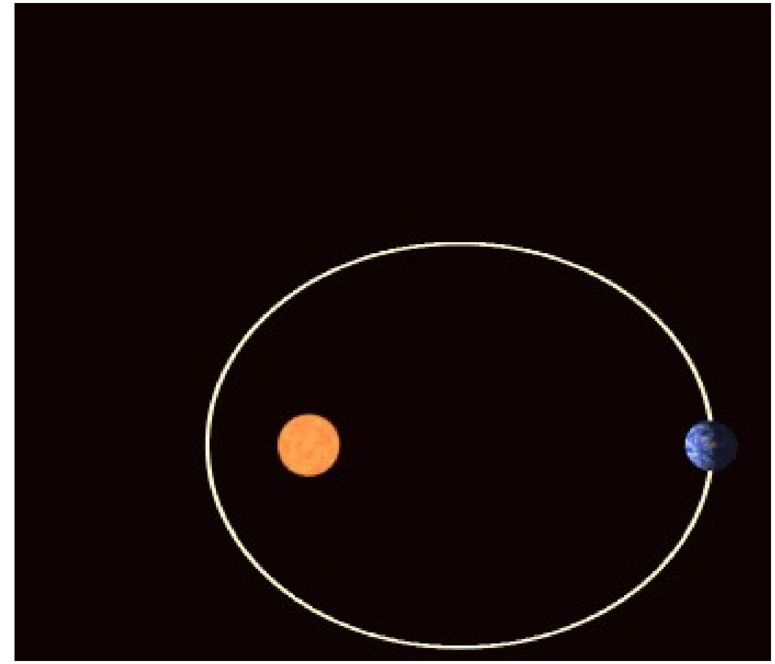
$$ds^2 = -\left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 + \left(c\, dt - a\sin^2\theta\, d\phi\right)^2 \frac{\Delta}{\rho^2} - \left(\left(r^2 + a^2\right) d\phi - ac\, dt\right)^2 \frac{\sin^2\theta}{\rho^2},$$

where $\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - R_S r + a^2 + R_Q^2$

- Inner and outer event horizon: $R_H^{\pm} = \frac{R_S}{2} \pm \sqrt{\frac{R_S^2}{4} - a^2 - R_Q^2}$

Four classic Solar System tests of GR

1. Perihelion precession of Mercury's orbit
2. Deflection of light by the Sun
3. Gravitational redshift of light
4. Gravitational (Shapiro) time delay

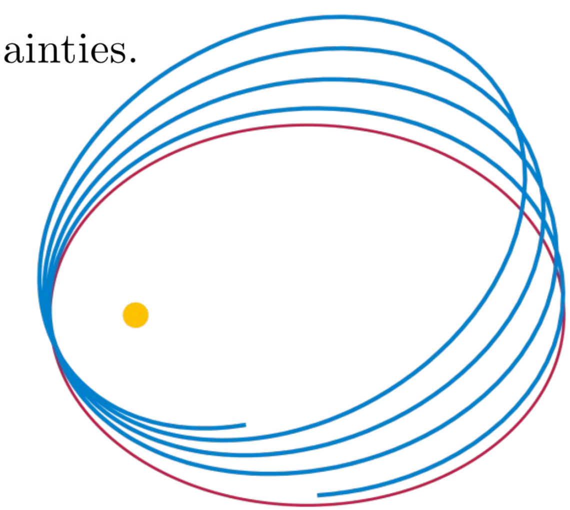


1. Bound orbits: precession of perihelia

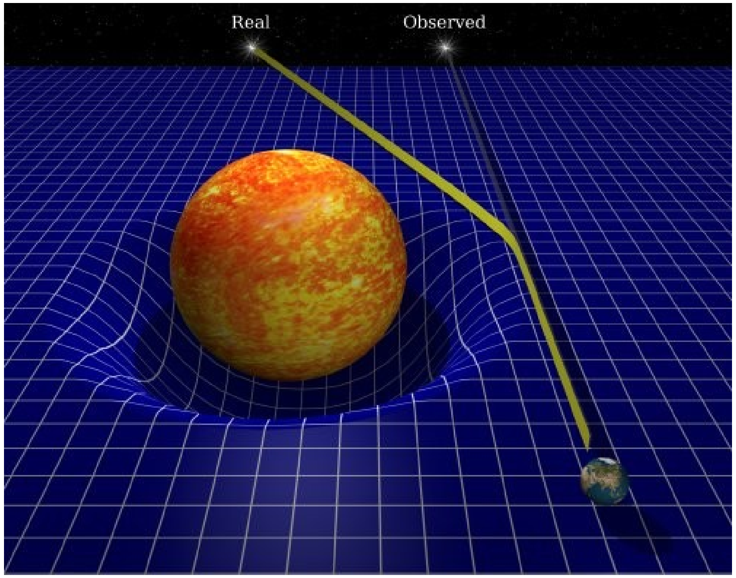
- In GR, a freely falling particle or photon can move along a bound or unbound orbit (geodesic) in the gravitational field of a central mass M
- Schwarzschild precession of bound elliptical orbits: $\Delta\varphi = \dfrac{6\pi GM}{c^2\, a\,(1-e^2)}$
- GR prediction for perihelion precession is confirmed with high accuracy by recent high-precision observations of the Solar System bodies and spacecrafts

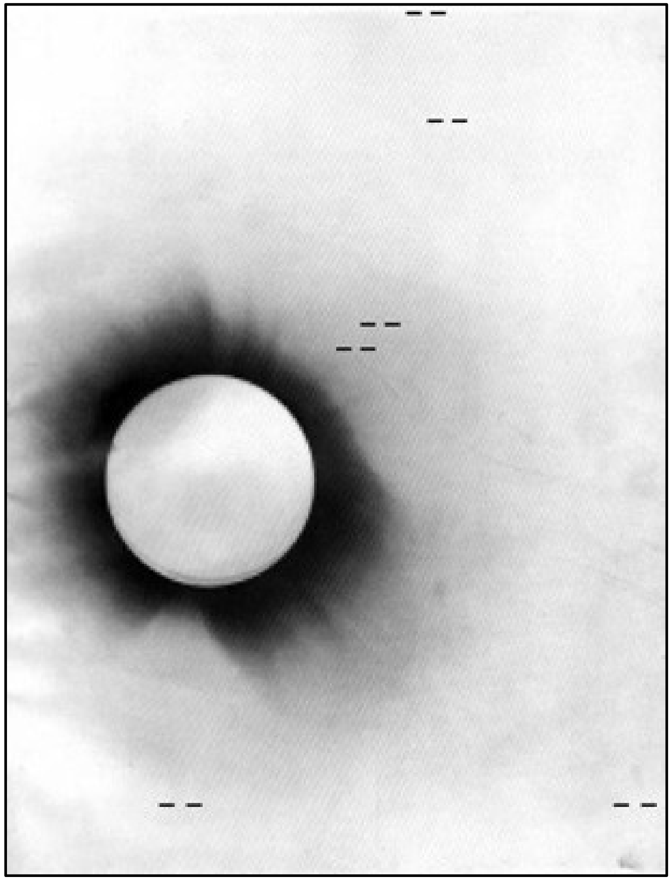
Corrections to the perihelion advances of planets ($''$/ cy) and their real uncertainties.

Mercury	Venus	Earth	Mars	Author
42.98	8.62	3.84	1.35	Brumberg, 1972
0.11±0.22	−3.03±0.71	−0.12±0.16	−0.35±0.24	Pitjeva, 1986
−0.017±0.052	—	—	—	Pitjeva, 1993
−0.0040±0.0050	0.024±0.033	0.006±0.007	−0.007±0.007	Pitjeva, 2009
Jupiter	**Saturn**	**Uranus**	**Neptune**	**Pluto**
0.067±0.093	−0.010±0.015	− 3.89±3.90	-4.44±5.40	2.84±4.51



2. Unbound orbits: deflection of light by the Sun



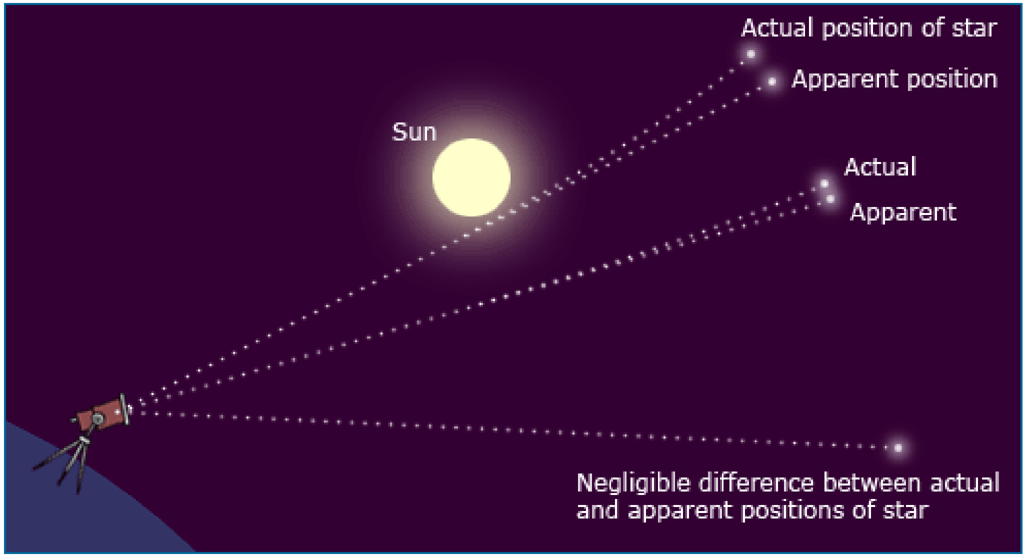


- Johann Georg von Soldner (1804) - trajectory of particle with speed c in Newtonian gravity deflected by angle:

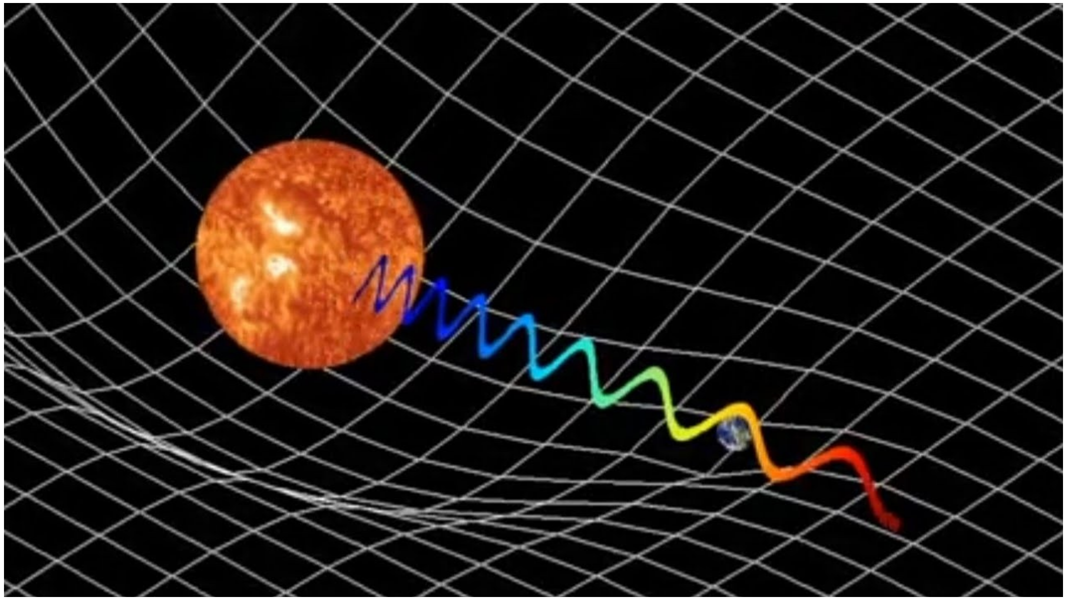
$$\alpha = \frac{2GM}{c^2\xi}$$

- Albert Einstein (1915) in GR - deflection angle of photons moving along geodesics: $\alpha = \frac{4GM}{c^2\xi}$

- Eddington - total solar eclipse in 1919:
  - No light bending: α = 0"
  - Newton's mechanics: α = 0".87
  - GR: α = 1".75
- Confirmation of Einstein's predictions:
  α_1 = 1".98 ± 0".12 α_2 = 1".61 ± 0".30
- **Gravitational lens** is a massive celestial object (or a distribution of matter), located between an observer and a distant background source, which gravitational force deflects the light rays from the source



3. Gravitational redshift of light



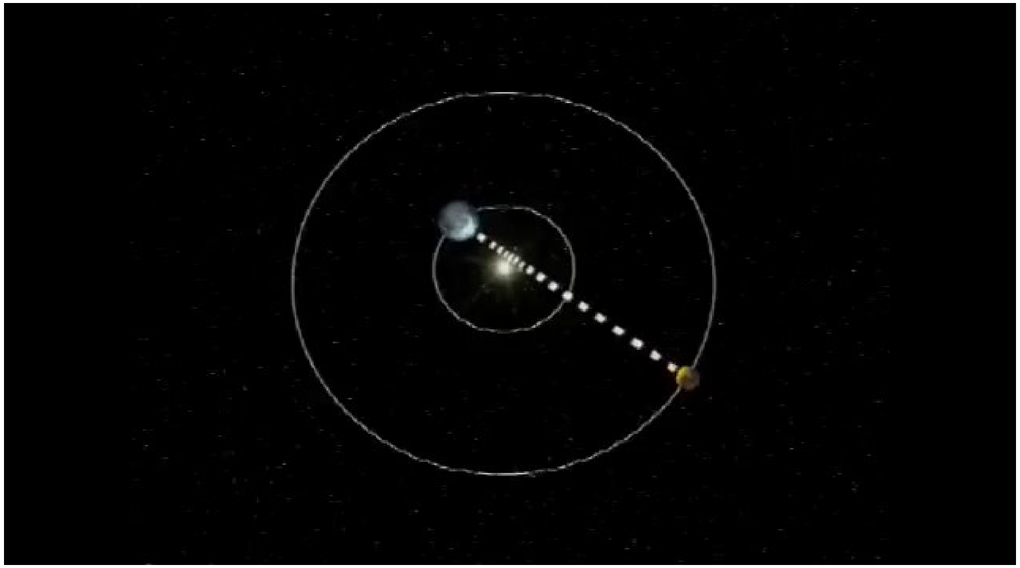
- Gravitational redshift z is the shift toward longer wavelengths of electromagnetic radiation emitted by a source in a gravitational field, e.g. at the surface of a massive star
- In spherically symmetric gravitational field, z is given by:

$$1 + z = \frac{\lambda_\infty}{\lambda_e} = \frac{1}{\sqrt{1 - \frac{R_S}{R_e}}}$$

  where λ_∞ is the wavelength of the light as measured by the observer at infinity, λ_e is the wavelength measured at the source of emission, and R_e is the radius at which the photon is emitted
- It was identified by astronomical observations in the spectral lines of the star Sirius B, white dwarf 40 Eridani B, Sun, sunlight reflected by the Moon and galaxy clusters
- Pound-Rebka experiment measured frequency shifts in gamma rays as they rose and fell in the gravitational field of the Earth
- Global Positioning System (GPS) must account for the gravitational redshift

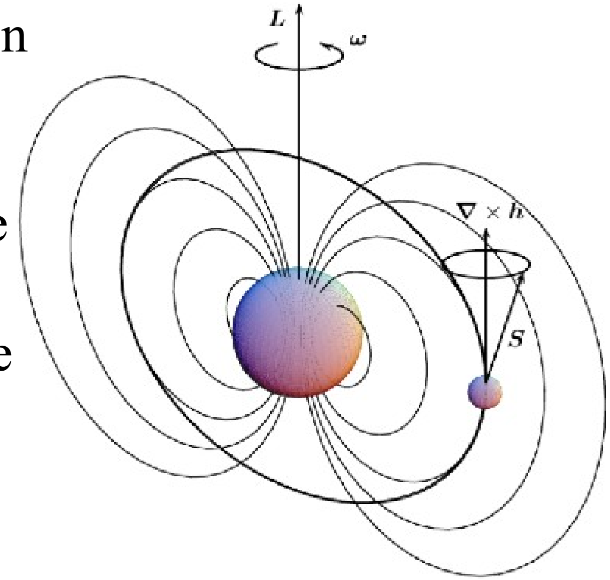
4. Shapiro time delay

- Radar signals passing near a massive object take slightly longer to travel to a target and longer to return than they would if the massive object was not present
- In the case of a nearly static gravitational field of moderate strength, such as in the case of stars and planets, **Shapiro time delay** is a special case of **gravitational time dilation** (difference of elapsed time between two events as measured by observers situated at varying distances from a gravitating mass)

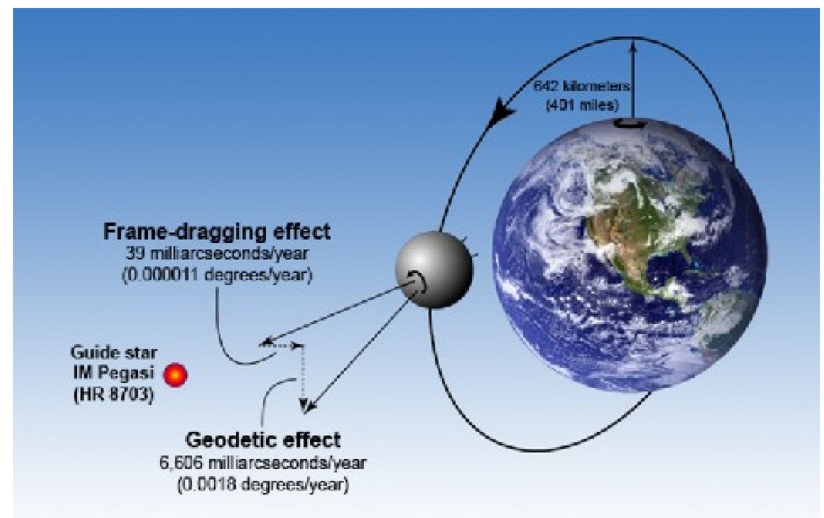


- Time delay Δt due to light traveling around a massive object is: $c\Delta t = -R_S \ln\left(1 - \vec{R} \cdot \vec{x}\right)$, where $\vec{R}$ is the unit vector pointing from the observer to the source, $\vec{x}$ is the unit vector pointing from the observer to the gravitating mass M, and the dot denotes the usual Euclidean scalar product
- During 1960s, Irwin I. Shapiro proposed and carried out measurements of the time required for radar signals to travel to Venus and Mercury and be reflected back to Earth
- Measured time delay Δt, due to the presence of the Sun, of a radar signal traveling from the Earth to Venus and back, was $\Delta t \sim 200$ μs, matching the time delay predicted by GR
- Shapiro time delay was also confirmed by ranging data of Voyager and Pioneer interplanetary spacecrafts

Precession of orbiting gyroscopes



- Gyroscope is a device used for measuring or maintaining orientation and angular velocity
- In Newtonian gravity, the spin axis of a perfect gyroscope orbiting the Earth would remain forever fixed with respect to absolute space
- In GR, the presence of a large rotating mass, such as Earth or Kerr black hole, causes spacetime to warp (curve) and twist, and thus the spin axis of a perfect gyroscope orbiting the central mass will precess with respect to the distant universe
- Spacetime warping and twisting both cause a precession (at ninety degree angles with respect to one another) of the gyroscopes orbiting the central mass
- The effect caused by warping of spacetime due to the presence of the central mass is called **de Sitter precession (geodetic effect)**
- The effect caused by twisting of spacetime (frame dragging) due to the rotation of the central mass is called **Lense-Thirring precession**
- The total precession is calculated by combining the de Sitter precession with the Lense-Thirring precession
- In 2011 **Gravity Probe B** mission gave direct proof of frame-dragging in the case of the Earth's rotation with a 19% margin of error



	Measured	Predicted
Geodetic precession (mas)	6602 ± 18	6606
Frame-dragging (mas)	37.2 ± 7.2	39.2

Exam questions

1. Vacuum solutions to the field equations: Schwarzschild, Kerr, Reissner-Nordström and Kerr-Newman metric
2. Classic Solar System tests of GR (perihelion precession of Mercury's orbit, deflection of light by the Sun, gravitational redshift of light, Shapiro time delay) and precession of orbiting gyroscopes (Lense-Thirring effect)

Literature

- Weinberg, S., 1972, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley-VCH
- Sean M. Carroll, 1997. *Lecture Notes on General Relativity*, arXiv, gr-qc/9712019