

MASS 2026 Course:
Gravitation and Cosmology

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Lecture 10

- **Distance measures in cosmology:**
 - Proper distance
 - * Hubble-Lemaître law
 - Comoving distance
 - Standard rulers and candles
 - * Angular diameter distance
 - * Luminosity distance
 - * Etherington's distance-duality relation
 - Light travel (or lookback) time
 - Comoving volume and intersection probability
 - Exercises

Distance measures in cosmology

- Formula for conversion of cosmological redshift z into distance D ?
- How to define D in expanding Universe?
- Answer: D may be defined in different ways, depending on how it is measured

- **FLRW metric:**

$$ds^2 = c^2 dt^2 - R^2(t) [d\chi^2 + S_k^2(\chi) d\Omega^2], \quad r = S_k(\chi), \quad S_k(\chi) = \begin{cases} \sin \chi, & k = +1 \\ \chi, & k = 0 \\ \sinh \chi, & k = -1 \end{cases}$$

- r, χ - dimensionless comoving radial coordinates

- 1. Proper distance between two objects: $D_P = R(t) \chi$**

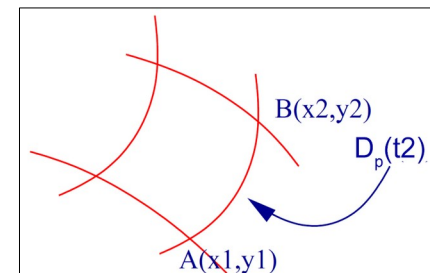
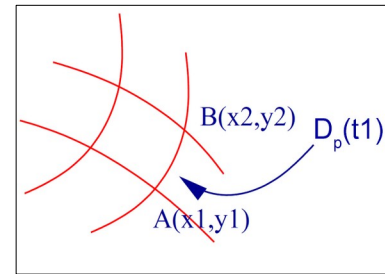
- distance which would be measured with rulers at the time of observation
- increasing with time

- 2. Comoving distance between two objects: $D_C = R_0 \chi$**

- equals to proper distance times the relative scale factor of the Universe now to then, or $(1 + z)$:

$$D_C = \frac{R_0}{R(t)} D_P = \frac{D_P}{a(t)} = (1 + z) D_P$$

- constant over time
- D_C is equal to D_P at the present time



Proper distance: Hubble-Lemaître law

- $R(t)$ represents the relative expansion of the Universe
- $R(t)$ relates the **proper distance** between a pair of objects, moving with the Hubble flow in an expanding or contracting Universe:

$$D(t) = \frac{R(t)}{R(t_0)} D(t_0)$$

- Lemaître, 1927, ASSB, 47, 49 (translated and reprinted in 1931, MNRAS, 91, 483):

$$V_r = \dot{D} = \frac{\dot{R}}{R_0} D_0 = \frac{\dot{R}}{R} D = H \cdot D$$

- z due to Doppler effect:

$$z = \frac{V_r}{c} \Rightarrow D(z) = \frac{c}{H} z$$

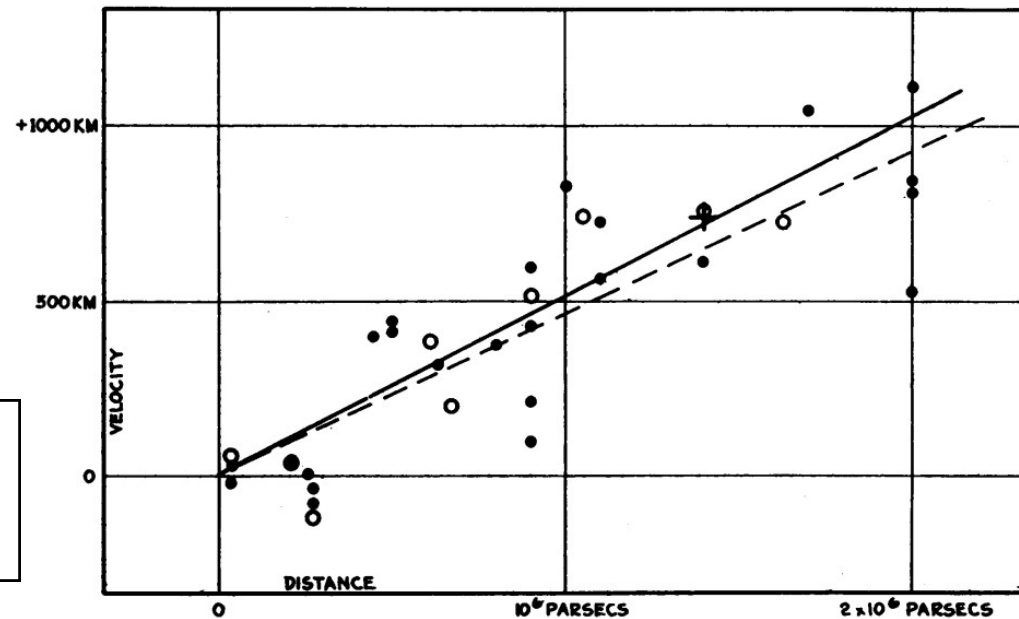


FIGURE 1

Velocity-Distance Relation among Extra-Galactic Nebulae.

Hubble, 1929, PNAS, 15, 168

Line-of-sight comoving distance

- Derivation from FLRW metric: $ds^2 = c^2 dt^2 - R^2(t) [d\chi^2 + S_k^2(\chi)d\Omega^2]$
- For photons: $ds = 0$, and for their radial tracks: $d\theta = d\varphi = 0$, so:

$$ds = 0 \wedge d\Omega = 0 \Rightarrow c dt = R(t) d\chi \Rightarrow d\chi = c \frac{dt}{R(t)} \Rightarrow \boxed{\chi(t) = c \int_t^{t_0} \frac{dt'}{R(t')}}$$

- On the other hand:

$$1 + z = \frac{R_0}{R} \Rightarrow \dot{z} = -R_0 \frac{\dot{R}}{R^2} \Leftrightarrow \frac{dz}{dt} = -\frac{R_0}{R} H \Leftrightarrow \boxed{\frac{R_0}{R(t)} dt = -\frac{dz}{H(z)}}$$

$$D_C(t) = R_0 \chi(t) = c \int_t^{t_0} \frac{R_0}{R(t')} dt' \Leftrightarrow \boxed{D_C(z) = c \int_0^z \frac{dz'}{H(z')}}$$

- $H(z)$ is given by the 1st Friedmann equation for Λ CDM model, and then:

- **Line-of-sight comoving distance:**

$$\boxed{D_C(z) = D_H \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_\kappa(1+z')^2 + \Omega_\Lambda}}}, \quad D_H = \frac{c}{H_0}$$

- Theoretical cosmological distance along radial coordinate in FLRW metric

Transverse comoving distance

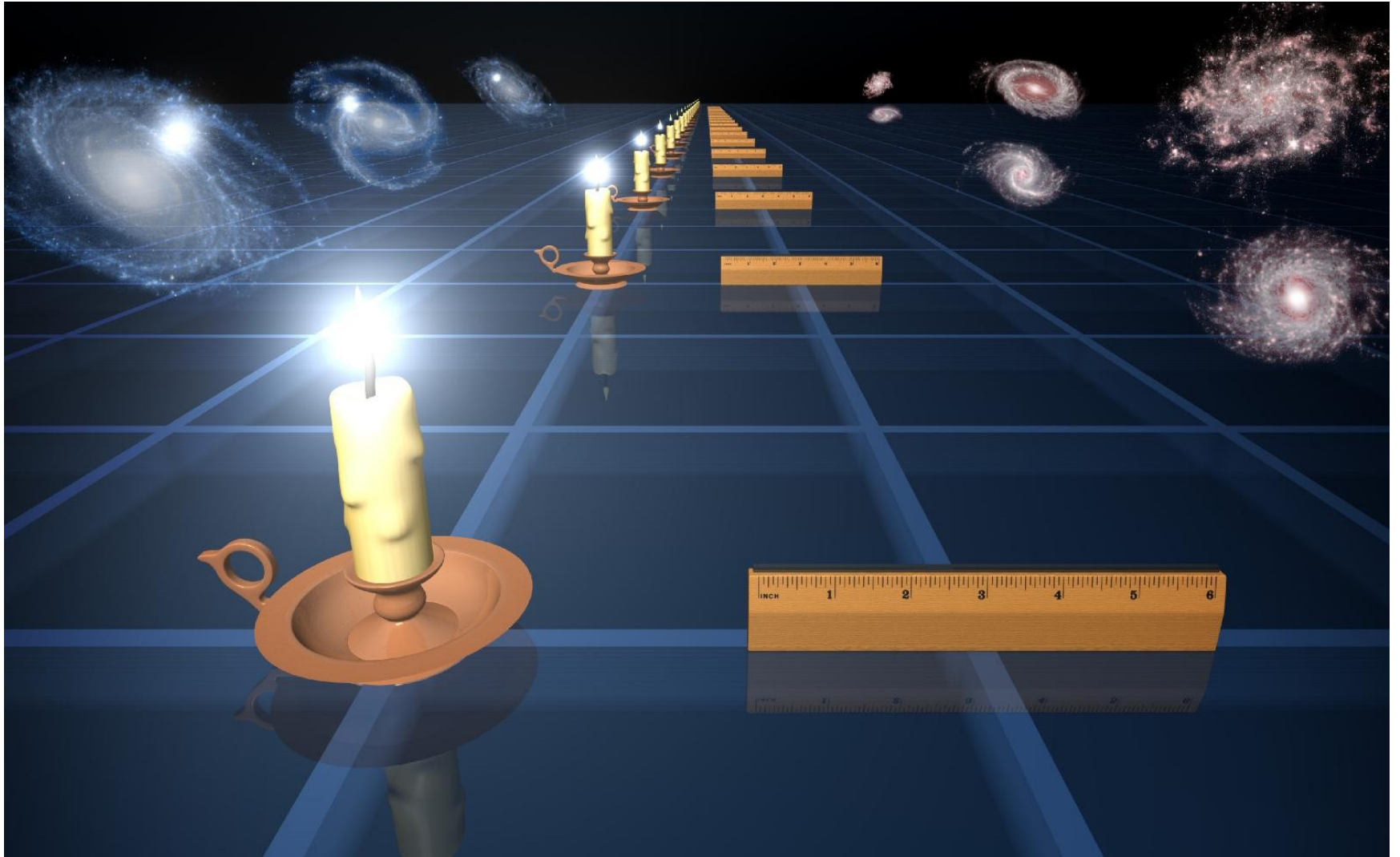
- If $\delta\theta$ is an angle between two objects at the same z , then the distance between them is: $\delta\theta D_M(z)$, where $D_M(z)$ is their **transverse comoving distance**:

$$D_M(z) = \begin{cases} D_H \frac{1}{\sqrt{\Omega_\kappa}} \sinh[\sqrt{\Omega_\kappa} D_C(z)/D_H], & \Omega_\kappa > 0 \\ D_C(z), & \Omega_\kappa = 0 \\ D_H \frac{1}{\sqrt{|\Omega_\kappa|}} \sin[\sqrt{|\Omega_\kappa|} D_C(z)/D_H], & \Omega_\kappa < 0 \end{cases}$$

- Analytic solution:

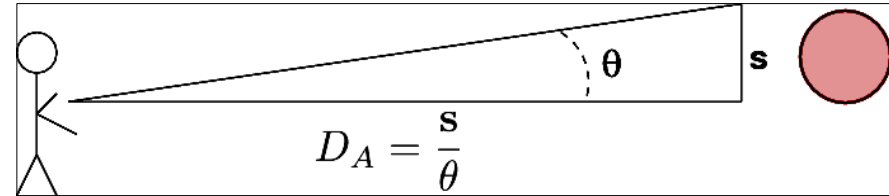
$$D_M(z) = D_H \frac{2 \left[2 - \Omega_M(1 - z) - (2 - \Omega_M)\sqrt{1 + \Omega_M z} \right]}{\Omega_M^2(1 + z)}, \quad \Omega_\Lambda = 0$$

Measuring distances by standard rulers and standard candles



Angular diameter distance

- D_A is Measured by standard cosmological rulers: objects which linear diameter s is known and does not change with cosmological time
- Defined as the ratio of an object's physical transverse size s to its observed angular size θ (in radians)



- **Calculating D_A from z :**

$$D_A(z) = \frac{D_M(z)}{1+z}$$

- Angular diameter distance between two objects at redshifts z_1 and z_2 :

$$D_{A12} = \frac{1}{1+z_2} \left[D_{M2} \sqrt{1 + \Omega_\kappa \frac{D_{M1}^2}{D_H^2}} - D_{M1} \sqrt{1 + \Omega_\kappa \frac{D_{M2}^2}{D_H^2}} \right]$$

where D_{M1} and D_{M2} are the corresponding comoving distances

- D_A is often used in the theory of gravitational lensing

Luminosity distance

- D_L is obtained from observed flux f of standard cosmological candles: objects which luminosity L is known and does not change with cosmological time

- D_L is used in the case of SN Ia

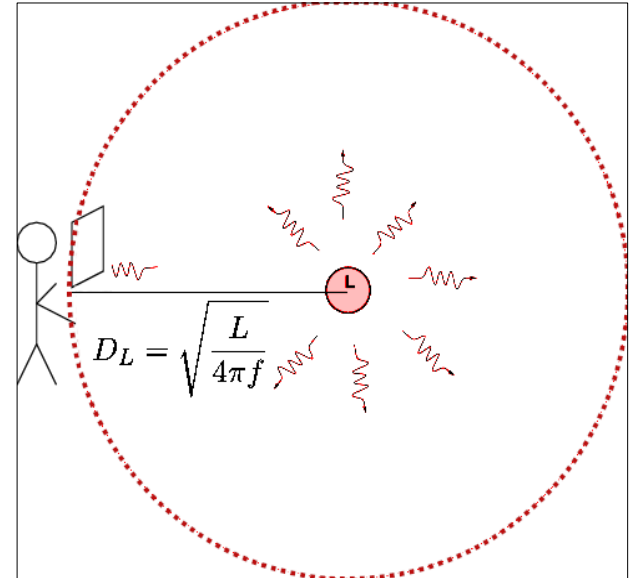
- **Calculating D_L from z :**

$$D_L(z) = (1 + z)D_M(z)$$

- **Etherington's distance-duality relation:**

$$D_L(z) = (1 + z)^2 D_A(z)$$

- Valid for all cosmological models based on the Riemannian geometry
- Using inappropriate D causes errors which increase with z
- Typical mistake: calculating L from f using proper distance obtained by Hubble's law (instead of D_L)



Light-travel distance

- **Light-travel (lookback) time:** time that it took light to reach the observer
- Difference between the age of the universe at the time of observation and its age at the time when the light was emitted:

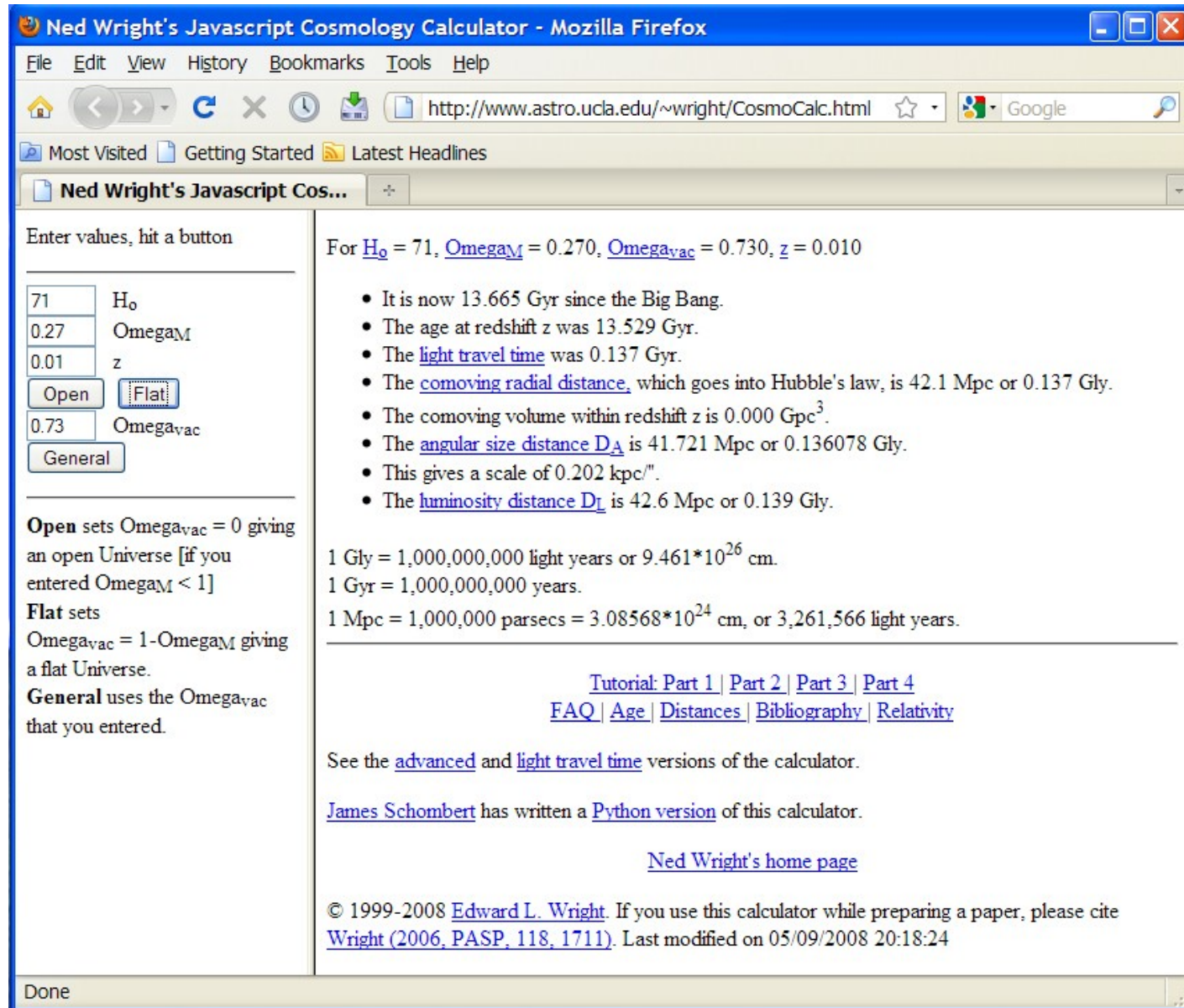
$$t_L(z) = t_H \int_0^z \frac{dz'}{(1+z')E(z')}$$

where $E(z) = \frac{H(z)}{H_0} = \sqrt{\Omega_M(1+z')^3 + \Omega_\kappa(1+z')^2 + \Omega_\Lambda}$

- Corresponding distance is: $D_T(z) = c \cdot t_L(z)$
- t_L is used for time delays between images of gravitational lenses

Numerical comparison of cosmological distances

I



The screenshot shows a web browser window titled "Ned Wright's Javascript Cosmology Calculator - Mozilla Firefox". The address bar shows the URL "http://www.astro.ucla.edu/~wright/CosmoCalc.html". The page content is as follows:

Enter values, hit a button

71	H_0
0.27	Ω_M
0.01	z
0.73	Ω_{vac}

Buttons: Open, Flat, General

Open sets $\Omega_{vac} = 0$ giving an open Universe [if you entered $\Omega_M < 1$]
Flat sets $\Omega_{vac} = 1 - \Omega_M$ giving a flat Universe.
General uses the Ω_{vac} that you entered.

For $H_0 = 71$, $\Omega_M = 0.270$, $\Omega_{vac} = 0.730$, $z = 0.010$

- It is now 13.665 Gyr since the Big Bang.
- The age at redshift z was 13.529 Gyr.
- The [light travel time](#) was 0.137 Gyr.
- The [comoving radial distance](#), which goes into Hubble's law, is 42.1 Mpc or 0.137 Gly.
- The comoving volume within redshift z is 0.000 Gpc³.
- The [angular size distance \$D_A\$](#) is 41.721 Mpc or 0.136078 Gly.
- This gives a scale of 0.202 kpc".
- The [luminosity distance \$D_L\$](#) is 42.6 Mpc or 0.139 Gly.

1 Gly = 1,000,000,000 light years or 9.461×10^{26} cm.
1 Gyr = 1,000,000,000 years.
1 Mpc = 1,000,000 parsecs = 3.08568×10^{24} cm, or 3,261,566 light years.

[Tutorial: Part 1](#) | [Part 2](#) | [Part 3](#) | [Part 4](#)
[FAQ](#) | [Age](#) | [Distances](#) | [Bibliography](#) | [Relativity](#)

See the [advanced](#) and [light travel time](#) versions of the calculator.

[James Schombert](#) has written a [Python version](#) of this calculator.

[Ned Wright's home page](#)

© 1999-2008 [Edward L. Wright](#). If you use this calculator while preparing a paper, please cite [Wright \(2006, PASP, 118, 1711\)](#). Last modified on 05/09/2008 20:18:24

Done

Numerical comparison of cosmological distances

II

Ned Wright's Javascript Cosmology Calculator - Mozilla Firefox

File Edit View History Bookmarks Tools Help

http://www.astro.ucla.edu/~wright/CosmoCalc.html

Ned Wright's Javascript Cos...

Enter values, hit a button

71	H ₀
0.27	Omega _M
0.5	z
<input type="button" value="Open"/>	<input type="button" value="Flat"/>
0.73	Omega _{vac}
<input type="button" value="General"/>	

Open sets Omega_{vac} = 0 giving an open Universe [if you entered Omega_M < 1]
Flat sets Omega_{vac} = 1 - Omega_M giving a flat Universe.
General uses the Omega_{vac} that you entered.

For H₀ = 71, Omega_M = 0.270, Omega_{vac} = 0.730, z = 0.500

- It is now 13.665 Gyr since the Big Bang.
- The age at redshift z was 8.647 Gyr.
- The [light travel time](#) was 5.019 Gyr.
- The [comoving radial distance](#), which goes into Hubble's law, is 1881.7 Mpc or 6.137 Gly.
- The comoving volume within redshift z is 27.909 Gpc³.
- The [angular size distance D_A](#) is 1254.5 Mpc or 4.0916 Gly.
- This gives a scale of 6.082 kpc".
- The [luminosity distance D_L](#) is 2822.6 Mpc or 9.206 Gly.

1 Gly = 1,000,000,000 light years or 9.461*10²⁶ cm.
1 Gyr = 1,000,000,000 years.
1 Mpc = 1,000,000 parsecs = 3.08568*10²⁴ cm, or 3,261,566 light years.

[Tutorial: Part 1](#) | [Part 2](#) | [Part 3](#) | [Part 4](#)
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Numerical comparison of cosmological distances

III

The screenshot shows a web browser window titled "Ned Wright's Javascript Cosmology Calculator - Mozilla Firefox". The address bar shows the URL "http://www.astro.ucla.edu/~wright/CosmoCalc.html". The page content is as follows:

Enter values, hit a button

71	H ₀
0.27	Omega _M
1	z
<input type="button" value="Open"/>	<input type="button" value="Flat"/>
0.73	Omega _{vac}
<input type="button" value="General"/>	

Open sets Omega_{vac} = 0 giving an open Universe [if you entered Omega_M < 1]
Flat sets Omega_{vac} = 1 - Omega_M giving a flat Universe.
General uses the Omega_{vac} that you entered.

For H₀ = 71, Omega_M = 0.270, Omega_{vac} = 0.730, z = 1.000

- It is now 13.665 Gyr since the Big Bang.
- The age at redshift z was 5.935 Gyr.
- The [light travel time](#) was 7.731 Gyr.
- The [comoving radial distance](#), which goes into Hubble's law, is 3317.1 Mpc or 10.819 Gly.
- The comoving volume within redshift z is 152.884 Gpc³.
- The [angular size distance D_A](#) is 1658.5 Mpc or 5.4095 Gly.
- This gives a scale of 8.041 kpc".
- The [luminosity distance D_L](#) is 6634.2 Mpc or 21.638 Gly.

1 Gly = 1,000,000,000 light years or 9.461*10²⁶ cm.
1 Gyr = 1,000,000,000 years.
1 Mpc = 1,000,000 parsecs = 3.08568*10²⁴ cm, or 3,261,566 light years.

[Tutorial: Part 1](#) | [Part 2](#) | [Part 3](#) | [Part 4](#)
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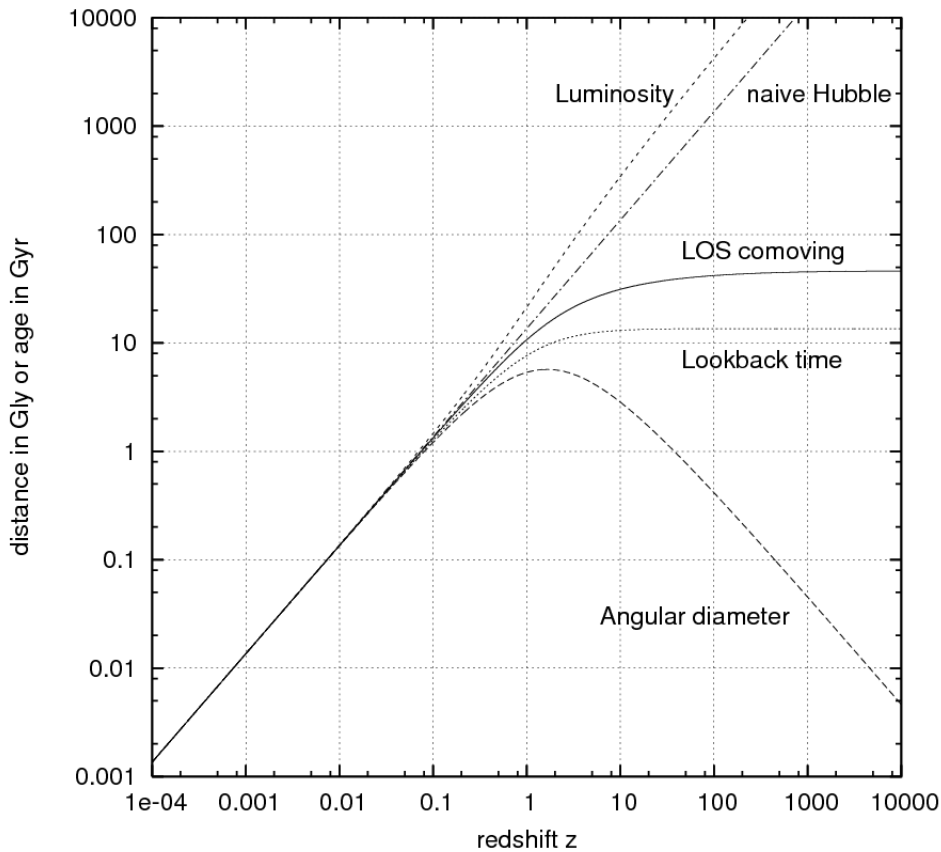
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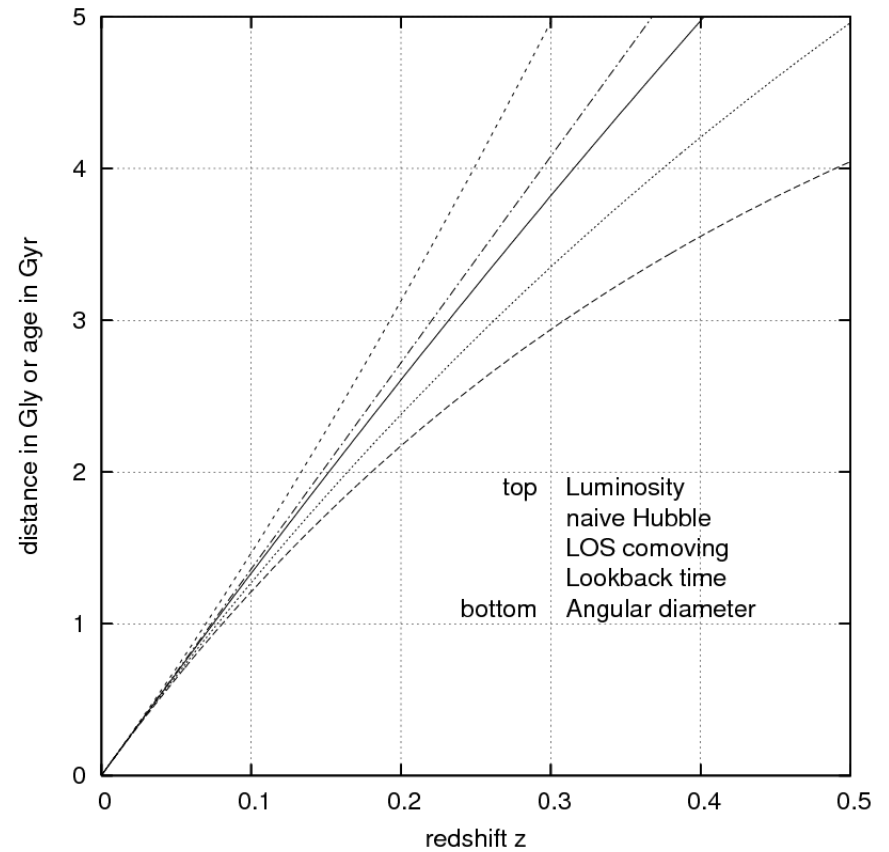
Done

Graphical comparison between cosmological distances

Comparison of Distance Measures $0 < z < 10,000$



Comparison of Distance Measures $0 < z < 0.5$



$$H_0 = 72 \text{ km/s/Mpc}, \Omega_M = 0.27, \Omega_\Lambda = 0.73$$

Comoving volume

- Volume in which number density of non-evolving objects locked into Hubble flow is constant with z
- Equals to proper volume times three factors of the relative scale factor now to then, or $(1 + z)^3$
- Element of comoving volume for solid angle $d\Omega$ and redshift interval dz :

$$dV_C(z) = D_H \frac{(1 + z)^2 D_A^2(z)}{E(z)} d\Omega dz$$

- Total all-sky comoving volume, out to redshift z :

$$V_C = \begin{cases} \left(\frac{4\pi D_H^3}{2\Omega_\kappa} \right) \left[\frac{D_M}{D_H} \sqrt{1 + \Omega_\kappa \frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_\kappa|}} \operatorname{arcsinh} \left(\sqrt{|\Omega_\kappa|} \frac{D_M}{D_H} \right) \right], & \Omega_\kappa > 0 \\ \frac{4\pi}{3} D_M^3, & \Omega_\kappa = 0 \\ \left(\frac{4\pi D_H^3}{2\Omega_\kappa} \right) \left[\frac{D_M}{D_H} \sqrt{1 + \Omega_\kappa \frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_\kappa|}} \operatorname{arcsin} \left(\sqrt{|\Omega_\kappa|} \frac{D_M}{D_H} \right) \right], & \Omega_\kappa < 0 \end{cases}$$

Intersection probability

- Given a population of objects with:
 1. $n(z)$ - comoving number density (number per unit volume) and
 2. $\sigma(z)$ - cross section (area)
- Differential probability that a line of sight will intersect one of the objects in redshift interval dz at redshift z :

$$dP(z) = n(z)\sigma(z)D_H \frac{(1+z)^2}{E(z)} dz$$

Exam question

1. Distance measures in cosmology

Literature

Articles:

1. David W. Hogg, 2000, *Distance measures in cosmology*, arXiv:astro-ph/9905116v4
2. Carroll, S. M., Press, W. H. & Turner, E. L. 1992, *The cosmological constant*, ARA&A, 30, 499
3. Davis, T. M. & Lineweaver, C. H. 2004, *Expanding Confusion: Common Misconceptions of Cosmological Horizons and the Superluminal Expansion of the Universe*, PASA, 21, 97 (arXiv:astro-ph/0310808)

Online cosmology calculators:

4. A list of cosmology calculators: http://ned.ipac.caltech.edu/he lp/cosmology_calc.html
5. Ned Wright's Javascript Cosmology Calculator: <http://www.astro.ucla.edu/~wright/CosmoCalc.html>

Exercise 1

Consider two flat ($\Omega_k = 0$), expanding universes: One that is matter dominated, i.e. $\Omega_M = 1$, and one that is dominated by a cosmological constant, i.e. $\Omega_\Lambda = 1$.

- a) Derive for both universes an expression for the comoving distance as a function of redshift z .
- b) The "observable universe" is the part of the universe up to redshift $z = \infty$. Compute for both of our universes the comoving volume of this region. Is there a qualitative difference between the two volumes?

Exercise 2

Assume a spatially flat universe, with $\Omega_{\text{tot}} = \Omega_M = 1$ and derive the formulas for angular diameter distance $D_A(z)$ and luminosity distance $D_L(z)$ in this cosmological model.

Exercise 3

Assume a flat ($\Omega_k = 0$) cosmological model with $H_0 = 71/\text{km/s/Mpc}$ and $\Omega_M = 0.27$, and graphically compare the following cosmological distances: $D_C(z)$, $D_A(z)$, $D_L(z)$ and naive Hubble distance: $D_H(z) = c z / H_0$ (see Hogg 2000 and Davis & Lineweaver 2004) .

Exercise 4

Write a Python script which computes angular diameter distance between two objects at redshifts z_1 and z_2 in the case of a flat ($\Omega_k = 0$) cosmological model with $H_0 = 71/\text{km/s/Mpc}$ and $\Omega_M = 0.27$, and use it to reconsider the exercise with an alien astronomer by calculating all three angular diameter distances: $D_A(0, z_a)$ - between you and an alien at redshift $z_a = 1$, $D_A(0, z_q)$ - between you and a quasar at $z_q = 2$ and $D_A(z_a, z_q)$ - between alien and quasar.

Solution 1

$$\text{a) } D_C(z) = D_H \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_\kappa(1+z')^2 + \Omega_\Lambda}}$$

$$1. \quad \Omega_M = 1, \Omega_\Lambda = 0, \Omega_\kappa = 0: \quad D_C(z) = D_H \int_0^z \frac{dz'}{\sqrt{(1+z')^3}} = 2D_H \left(1 - \frac{1}{\sqrt{1+z}}\right)$$

$$2. \quad \Omega_M = 0, \Omega_\Lambda = 1, \Omega_\kappa = 0: \quad D_C(z) = D_H \int_0^z dz' = D_H z$$

$$\text{b) } \Omega_\kappa = 0: \quad V_C(z) = \frac{4\pi}{3} D_M^3(z) = \frac{4\pi}{3} D_C^3(z)$$

$$1. \quad \Omega_M = 1, \Omega_\Lambda = 0: \quad V_C = \frac{32\pi}{3} D_H^3$$

$$2. \quad \Omega_M = 0, \Omega_\Lambda = 1: \quad V_C = \infty$$

- The volume of the observable Universe in the case of the cosmological constant is not finite, while it is finite in the case of a matter dominated universe

Solution 2

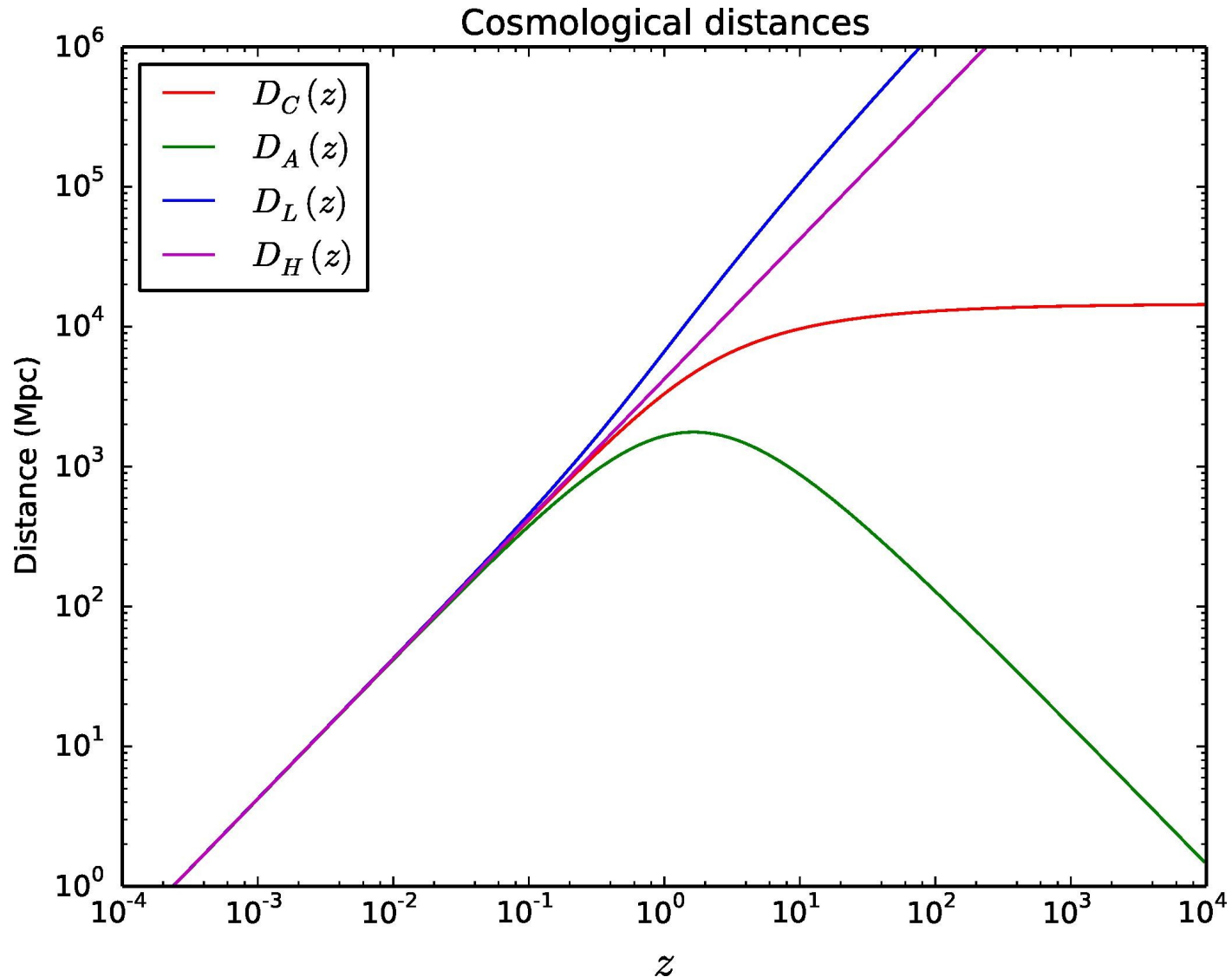
$$\Omega_\kappa = 0 \Rightarrow D_A(z) = \frac{D_M(z)}{1+z} = \frac{D_C(z)}{1+z}$$

$$\Omega_M = 1 \wedge \Omega_\Lambda = 0 \wedge \Omega_\kappa = 0 \Rightarrow D_C(z) = 2D_H \left(1 - \frac{1}{\sqrt{1+z}} \right) \Rightarrow$$

$$D_A(z) = \frac{2D_H}{1+z} \left(1 - \frac{1}{\sqrt{1+z}} \right)$$

$$D_L(z) = (1+z)^2 D_A(z) = 2D_H(1+z) \left(1 - \frac{1}{\sqrt{1+z}} \right)$$

Solution 3



Solution is obtained by Python script in "cosm_dist.py"

Solution 4

Output from Python script in "DA.py":

Enter the 1st redshift: 1.0

Enter the 2nd redshift: 2.0

Enter the Hubble constant: 71.0

Enter the Omega_matter: 0.27

Angular diameter distance $DA(0,z1) = 1658.7$ Mpc

Angular diameter distance $DA(0,z2) = 1748.4$ Mpc

Angular diameter distance $DA(z1,z2) = 642.6$ Mpc