MASS 2026 Course:

Gravitation and Cosmology

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Lecture 12

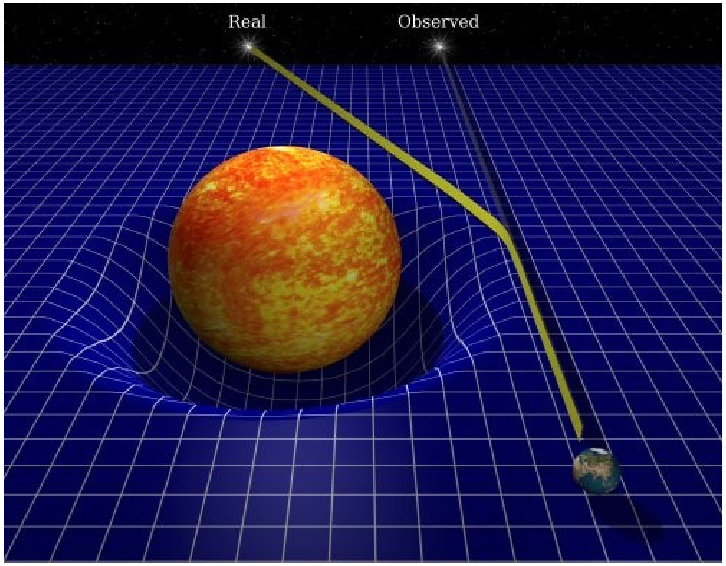
- Gravitational lenses:
  - Definition and types: **strong** (macro and micro) and **weak** lensing
  - Point-like lenses: deflection angle, lens equation and Einstein radius
  - Extended lenses: surface mass density, deflection (lensing) potential, shear and image distortions
- Cosmological applications of gravitational lensing:
  - Strong lenses as natural telescopes: detection of the most distant galaxies
  - Fermat potential and lensing time delay: determination of H_0
  - Optical depth and statistics of strong lenses: constraints on the cosmological parameters
  - Microlensing applications for studying physics and spacetime geometry in vicinity of supermassive black holes
  - Weak lensing and detection of dark matter by mass reconstruction
- Problems with standard ΛCDM cosmological model: Hubble tension and cosmological constant problem (vacuum catastrophe)
- Exercises

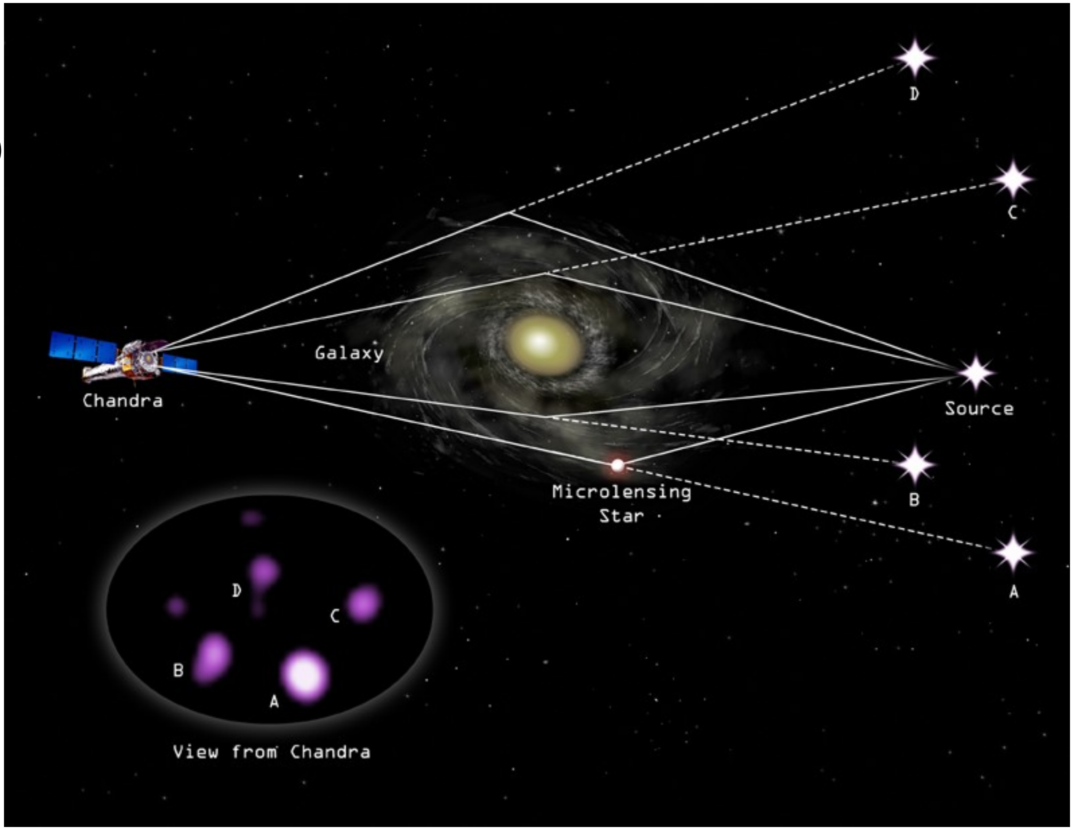
Gravitational lensing

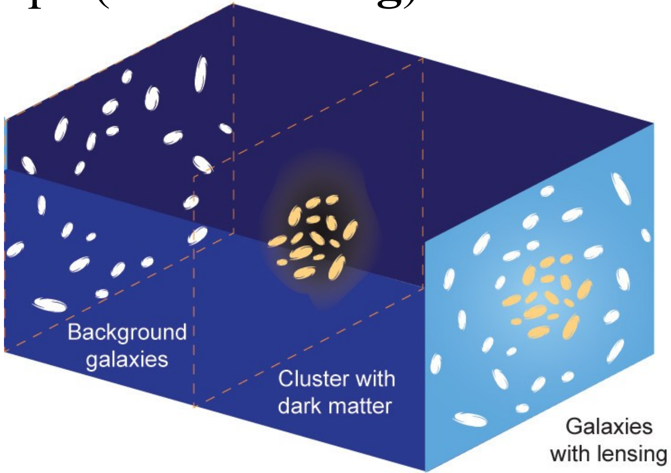
- Natural phenomenon caused by the bending of light in the gravitational field of a massive object
- GR: light travels along null geodesics and the **light deflection angle** is:

$$\alpha = \frac{4GM}{c^2\xi}$$

- **Gravitational lens** is a massive celestial object (or a distribution of matter), located between an observer and a distant background source, which gravitational force deflects the light rays from the source, producing the multiple images of the source (**macrolensing**) amplification of its brightness (**microlensing**), or distortion of its shape (**weak lensing**)

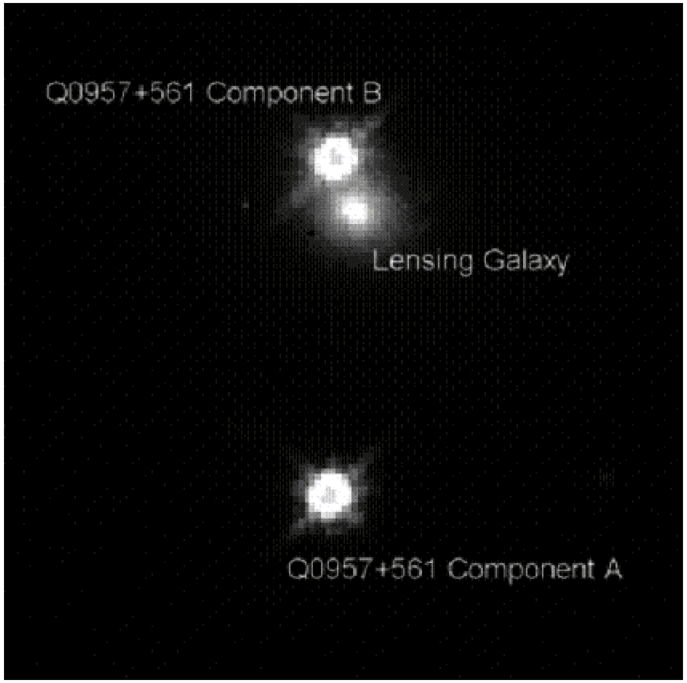


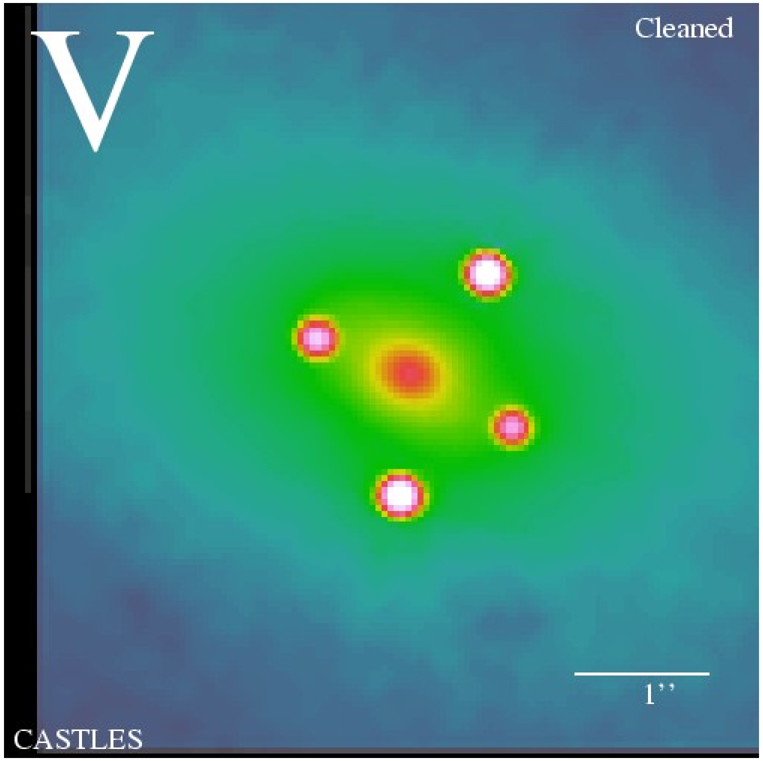




Macrolensing examples

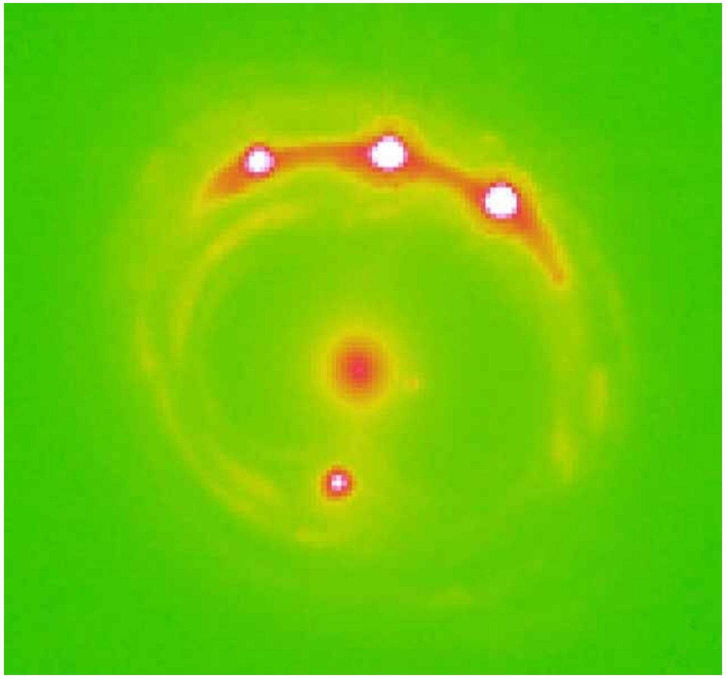
- Right: the first identified gravitational lens: double-imaged quasar QSO 0957+561, discovered by Dennis Walsh, Robert Carswell and Ray Weyman in 1979
- Both components have identical redshifts and spectra





Quasar Q2237+030 at z=1.695 (Einstein cross) and lensing galaxy ZW2237+030 at z=0.0394

- CASTLES Gravitational Lens Data Base: http://www.cfa.harvard.edu/castles/



Quasar RXJ1131-1231

Gravitational lensing types and applications

1. **Strong lensing:**
   - by galaxies (**macrolensing**) - multiple images of the background sources: determination of cosmological parameters (H_0 from time delays, Ω_M, Ω_Λ, Ω_k from lensing statistics)
   - by stars (**microlensing**) - amplification (magnification) of the background sources: detection of extrasolar planets, studying the innermost regions of active galaxies around their central supermassive black holes, constraining cosmological parameters
   - by clusters of galaxies - giant arcs as images of distant background galaxies: finding the most distant galaxies in the Universe (natural telescopes)
2. **Weak lensing:**
   - by foreground matter with lower density distribution - shape distortions of the background sources: the only direct mean to detect the dark matter, studying the distribution of visible and dark matter in the Universe

Gravitational lensing theory

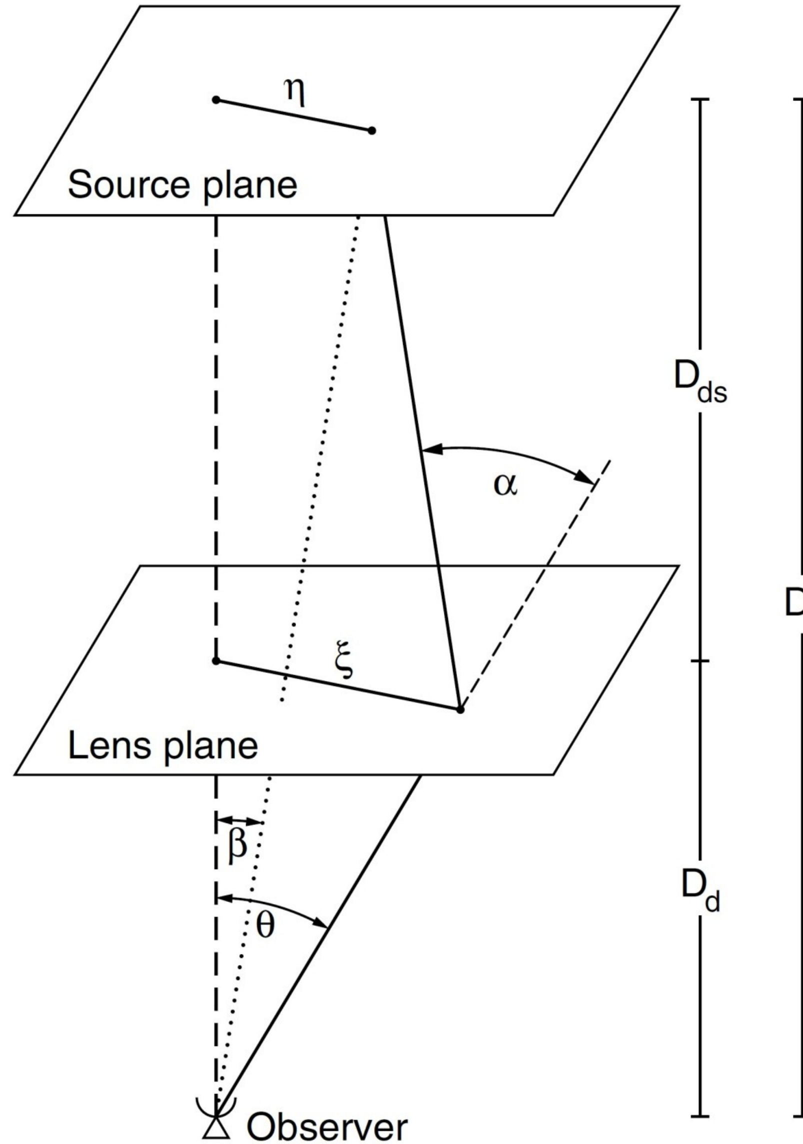
- **Geometrically thin lens:** the field equations of GR can be linearized if the gravitational field is weak (i.e. for the small deflection angle), and the ray can be approximated as a straight line near the deflecting mass
- **Lens equation** (see the figure):

$$\vec{\eta} = \frac{D_s}{D_d}\vec{\xi} - D_{ds}\hat{\vec{\alpha}}(\vec{\xi})$$

- **Light deflection angle**: $\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4GM}{c^2}\frac{\vec{\xi}}{\xi^2}$
- Angular coordinates $\vec{\beta}$ and $\vec{\theta}$, and **scaled (reduced) deflection angle** $\vec{\alpha}(\vec{\theta})$:

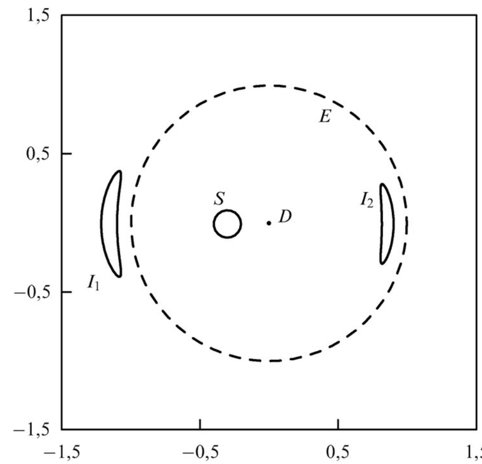
$$\vec{\eta} = D_s\vec{\beta},\ \vec{\xi} = D_d\vec{\theta},\ \vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s}\hat{\vec{\alpha}}(\vec{\theta}) \Rightarrow$$

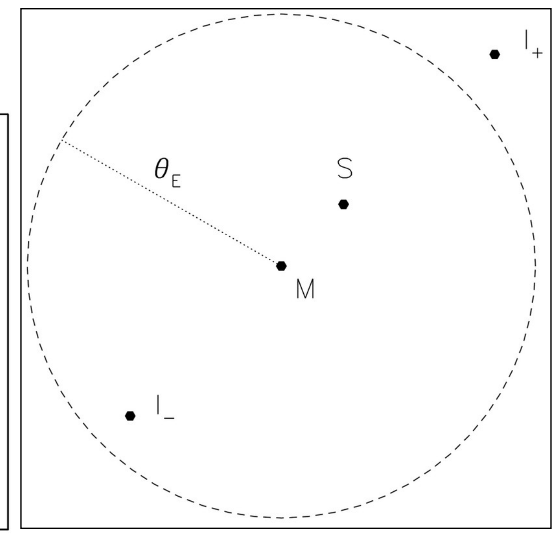
- Dimensionless lens equation: $\boxed{\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})}$
- Angular diameter distances between the observer and lens, observer and source, and lens and source: $D_d = D_A(0, z_d),\ D_s = D_A(0, z_s),\ D_{ds} = D_A(z_d, z_s)$



Einstein radius and point-like lens

- Solution of the lens equation for a point mass M and perfect alignment between the observer, lens and source $\vec{\eta} = \vec{\beta} = 0 \Rightarrow$
- **Linear Einstein radius** (in the lens plane)**:** $\xi_E = \sqrt{\frac{4GM}{c^2}\frac{D_d D_{ds}}{D_s}} \quad \Rightarrow$
- **Angular Einstein radius:** $\boxed{\theta_E = \frac{\xi_E}{D_d} = \sqrt{\frac{4GM}{c^2 D}}}$, where $D = \frac{D_d D_s}{D_{ds}}$ is effective lens distance
- Typical Einstein radius:
  - for a galaxy: on the order of 1"
  - for galaxy clusters: on the order of 10"
  - for a star: on the order of μas
- Separation between the images ≈ twice the average Einstein radius

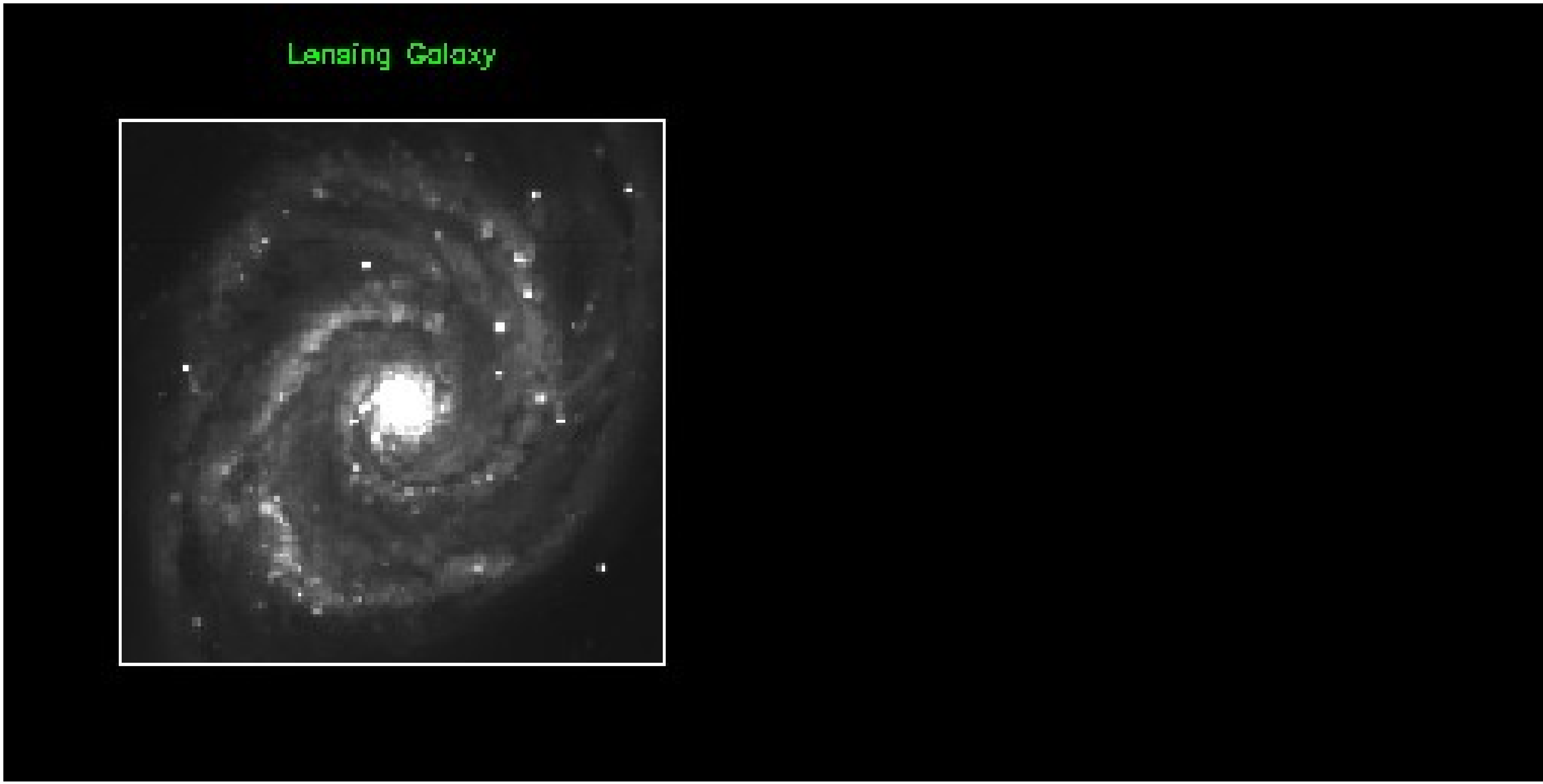




<u>Point-like lens</u>

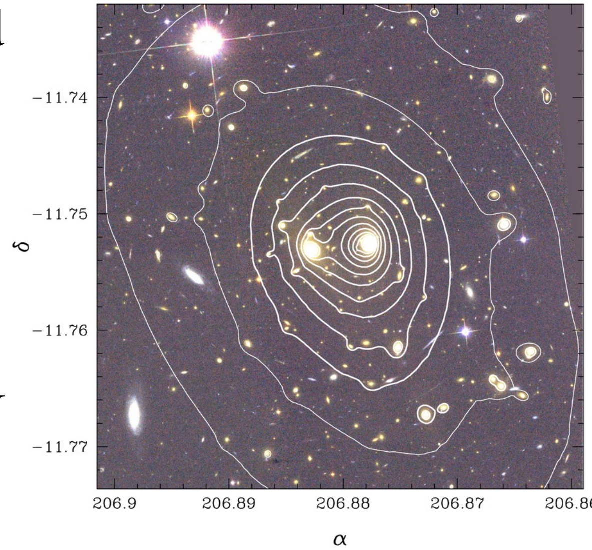
- The simplest lensing model that produces 2 mirror-inverted images:
- **Deflection angle:** $\vec{\alpha}(\vec{\theta}) = \theta_E^2 \frac{\vec{\theta}}{\theta^2} \quad \Rightarrow$
- **Lens equation:** $\vec{\beta} = \vec{\theta} - \theta_E^2 \frac{\vec{\theta}}{\theta^2} \Rightarrow$ **Image positions:** $\theta_{1,2} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$

Strong lensing by galaxies: point-like macrolens model



- Powerful method for measuring the masses of the lensing galaxies

Extended lenses and deflection potential



- 3D mass density $\rho(\vec{r})$ of an **extended lens** can be projected along the line of sight onto the lens plane to obtain the 2D **surface mass density** distribution: $\Sigma(\vec{\xi}) = \int_0^{D_s} \rho(\vec{r}) dz,$

  where $\vec{r}$ is a 3D vector in space, and $\vec{\xi}$ is a 2D vector in the lens plane
- **Critical surface mass density** is given by the lens mass M "smeared out" over the area of the Einstein ring:

$$\Sigma_{cr} = \frac{M}{\pi \xi_E^2} = \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}} = 0.35 \,\mathrm{g\,cm}^{-2} \left(\frac{D}{1\,\mathrm{Gpc}} \right)^{-1}$$

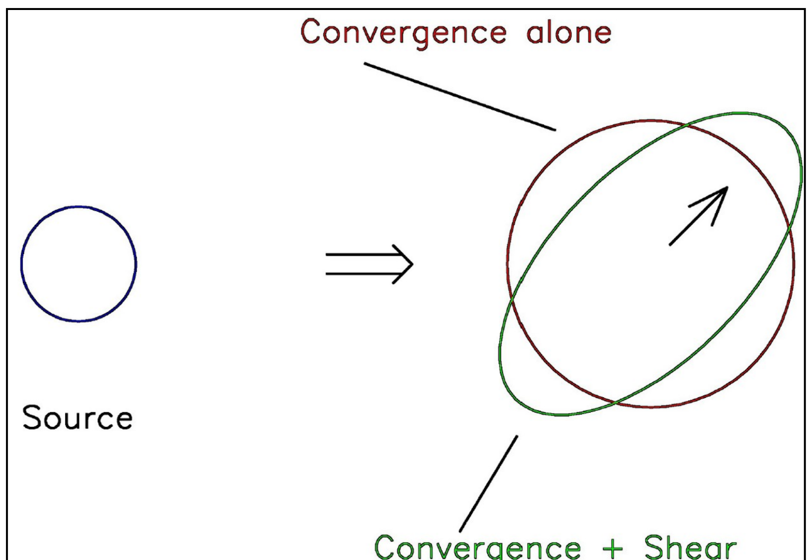
- **Dimensionless surface mass density (convergence) *κ*:** $\kappa(\vec{\theta}) := \frac{\Sigma(D_d \vec{\theta})}{\Sigma_{cr}}$
- A mass distribution for which $\kappa \geq 1$, i.e. $\Sigma \geq \Sigma_{cr}$ produces multiple images for some source positions (limit between "weak" from "strong" lenses)
- **Deflection potential *ψ*:** $\psi(\vec{\theta}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \kappa(\vec{\theta}') \ln |\vec{\theta} - \vec{\theta}'| d^2\theta'$
- Deflection angle as a gradient of *ψ*: $\vec{\alpha} = \vec{\nabla}\psi(\vec{\theta}) \Rightarrow$ lens equation: $\boxed{\vec{\beta} = \vec{\theta} - \vec{\nabla}\psi(\vec{\theta})}$
- **Poisson equation:** $\nabla^2 \Phi = 4\pi G \rho \Rightarrow \nabla^2 \psi = 2\kappa$
- Different surface mass distributions ⇒ different deflection potentials ⇒ different extended lens models ⇒ different number of images (3, 4, 5, … images of a source)

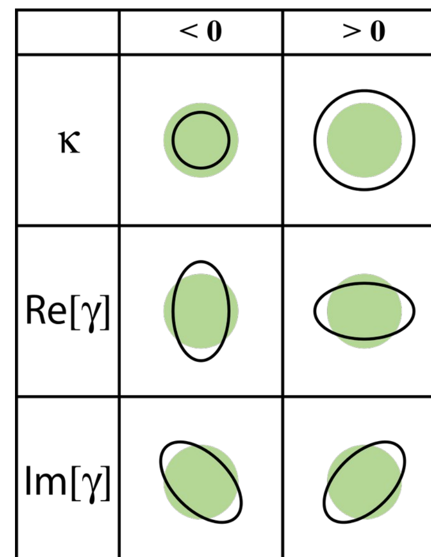
Shear and image distortions

- Expansion of the deflection potential in a Taylor series:

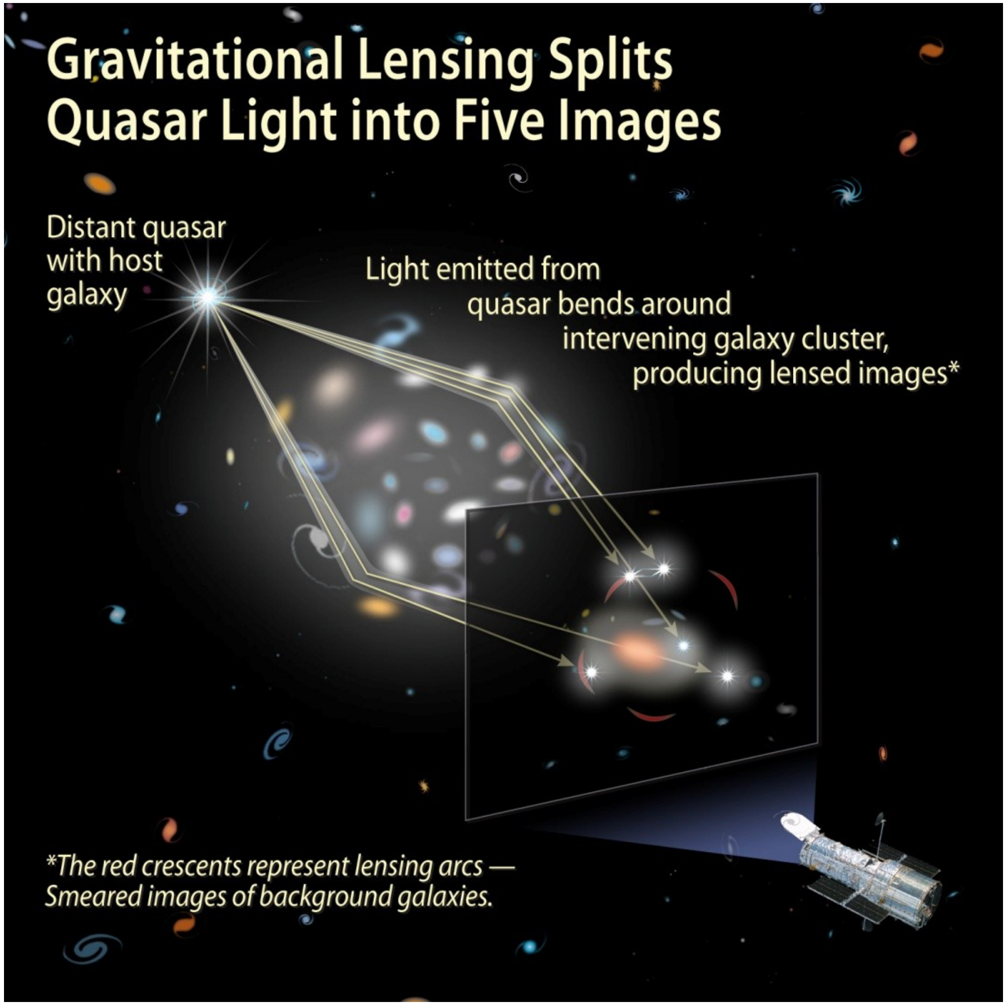
$$\psi(x,y) = \psi_0 + \psi_x x + \psi_y y + \frac{1}{2}\left(\psi_{xx}x^2 + \psi_{yy}y^2\right) + \psi_{xy}xy + \ldots$$

- $\psi_0 = 0$ since constant terms in the potential have no effect on lensing observables
- The linear terms are constant components of the reduced deflection angle
- <u>**Definitions:**</u> $\kappa \equiv \frac{1}{2}(\psi_{xx} + \psi_{yy}), \quad \gamma_1 \equiv \frac{1}{2}(\psi_{xx} - \psi_{yy})$ and $\gamma_2 \equiv \psi_{xy}$
- κ is **convergence**, and γ is **shear:** $(\gamma_1, \gamma_2) \equiv (\gamma \cos 2\theta_\gamma, \gamma \sin 2\theta_\gamma)$
- γ_1 and γ_2 are the shear components, while γ and θ_γ are its amplitude and direction
- The lensed images are distorted in shape and size:
  - Size distortion is due to the convergence κ: isotropic magnification
  - Shape distortion is due to the tidal gravitational field described by the shear γ

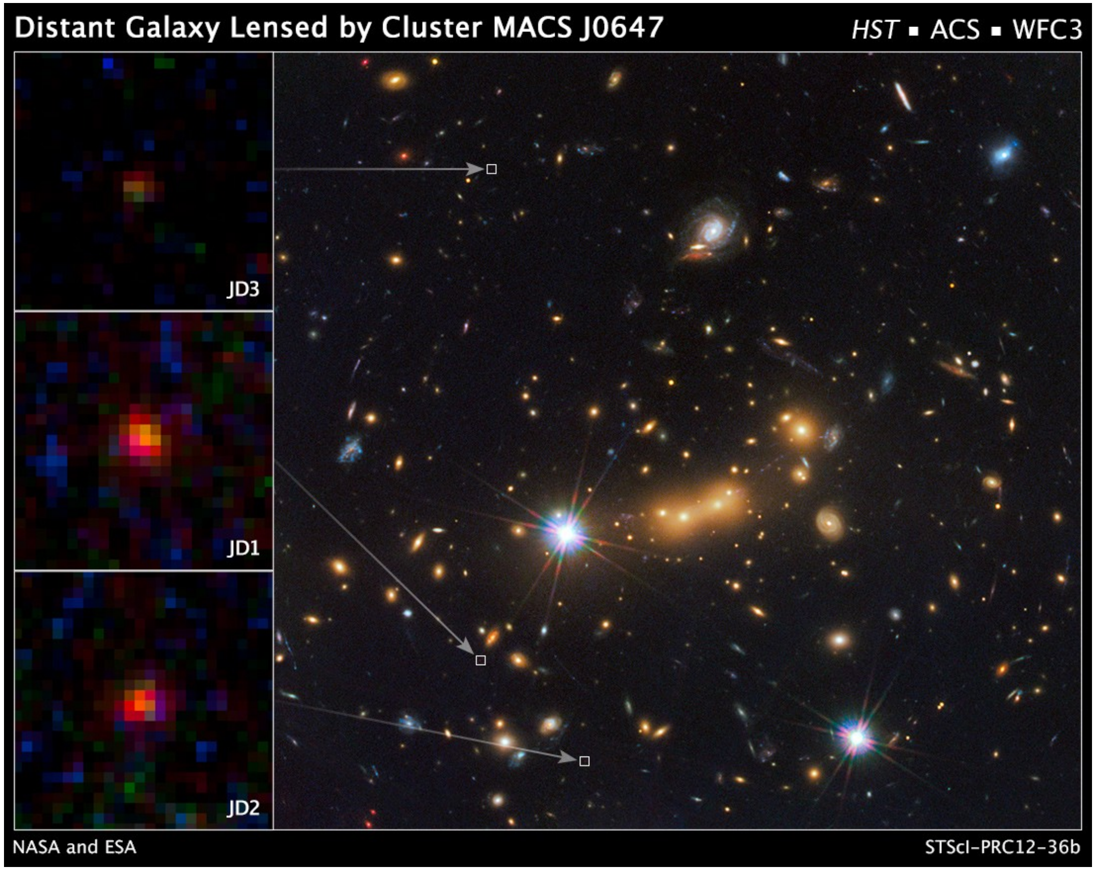




Strong lensing by galaxy clusters (natural telescopes): finding the most distant galaxies

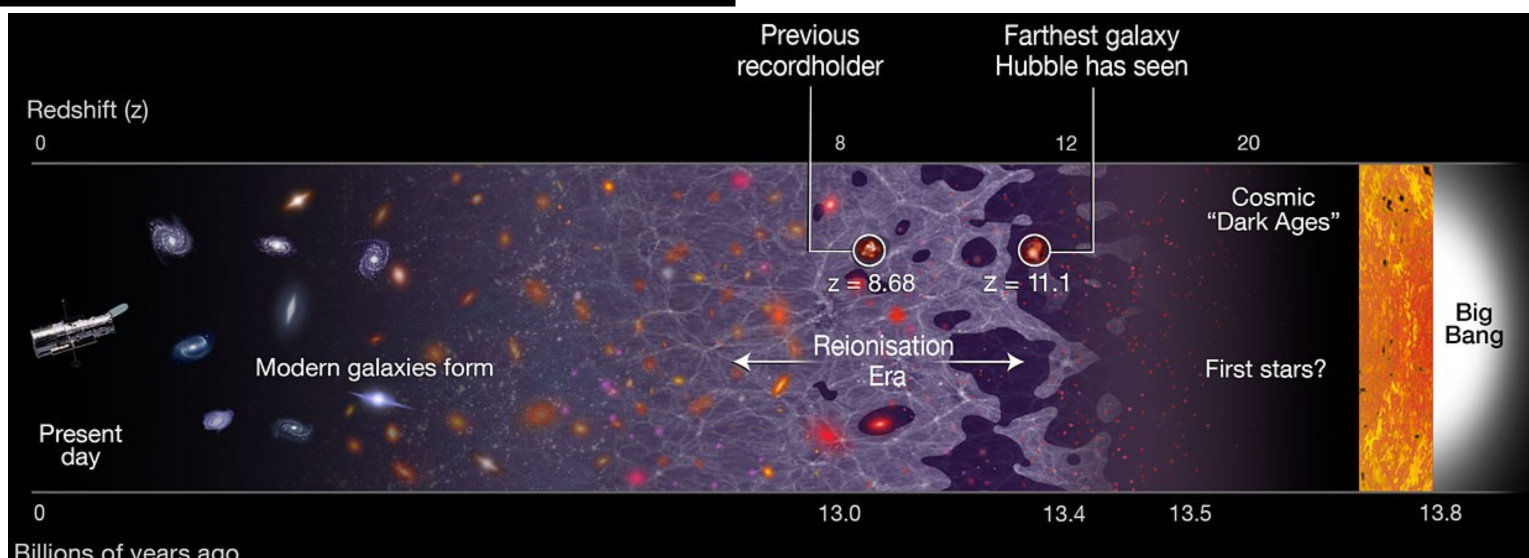


Right: red arc and point as images of the most distant galaxy known until 2004, located at $z \sim 7$ ($\approx 13 \times 10^9$ ly)

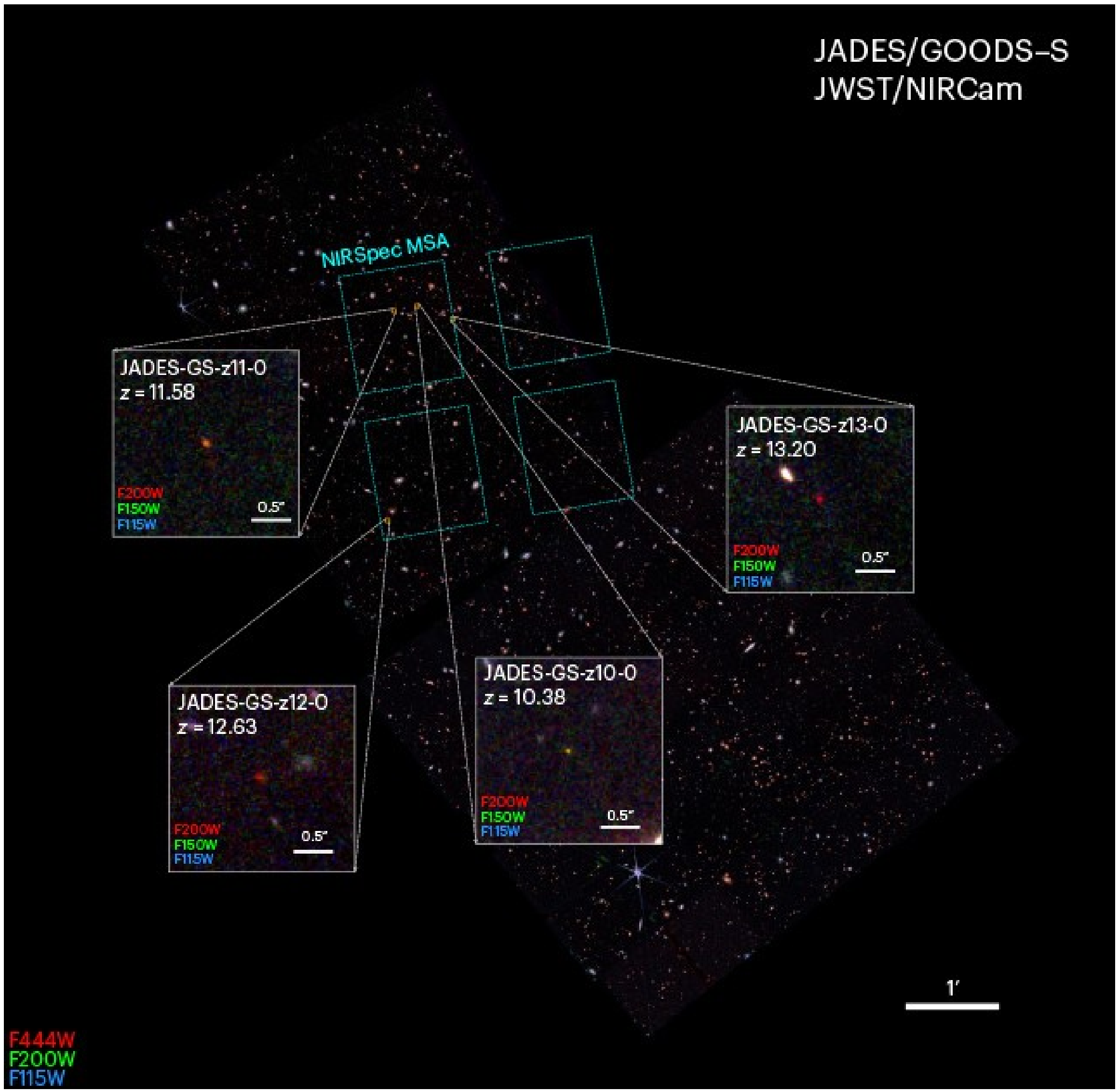


MACS0647-JD: the most distant galaxy discovered in 2012: the distance to the cluster is 5.6 billion ly (z = 0.591) and to the lensed galaxy is 13.3 billion ly (z = 11)

- Distance record from 2016: galaxy GN-z11 at z = 11.1



Four most distant galaxies ever seen detected by JWST in 2023



Gravitational lensing by galaxy cluster Abell 2744 (Pandora's Cluster)

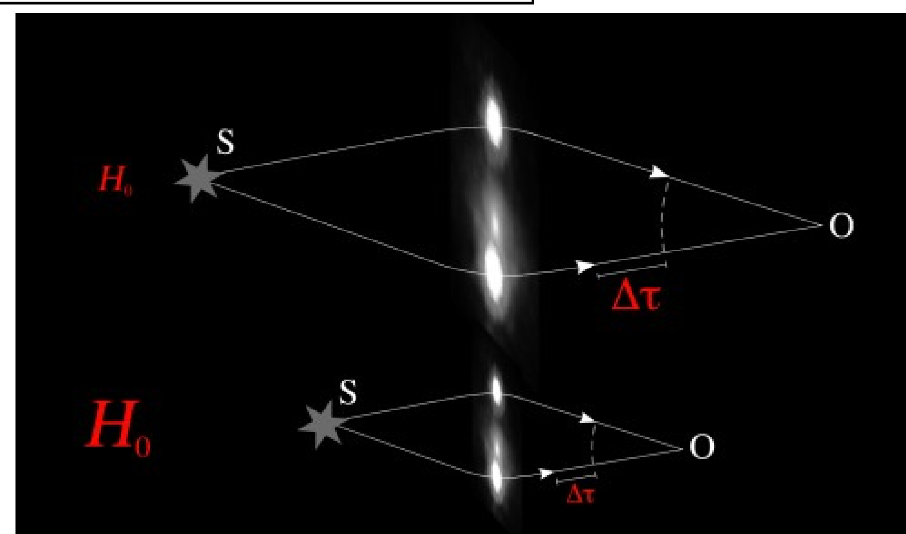
Robertson et al. 2023, Nature Astronomy, 7, 611.

Fermat potential and lensing time delay

- **Fermat principle:** physical light rays take the paths that make their light-travel time stationary
- Lens equation can be rewritten as: $\vec{\nabla}\left[\frac{1}{2}(\vec{\theta}-\vec{\beta})^2-\psi(\vec{\theta})\right]=0 \Leftrightarrow \vec{\nabla}\tau(\vec{\theta},\vec{\beta})=0$
- **Fermat potential**: $\tau(\vec{\theta},\vec{\beta})=\frac{1}{2}(\vec{\theta}-\vec{\beta})^2-\psi(\vec{\theta})$ is, up to an affine transformation, the **light travel time** along a ray starting at position $\vec{\beta}$, traversing the lens plane at position $\vec{\theta}$ and arriving at the observer
- **Light travel time (**physical time delay function**)** for a lensed image:

$$\tau(\vec{\theta},\vec{\beta})=\frac{D_{\Delta t}}{c}\left(\frac{1}{2}(\vec{\theta}-\vec{\beta})^2-\psi(\vec{\theta})\right), \quad D_{\Delta t}=(1+z_d)\frac{D_d D_s}{D_{ds}},$$

- **Time-delay distance** $D_{\Delta t}\propto H_0^{-1}$ and very weakly sensitive to Ω_M and Ω_Λ
- **Lensing time delay**: $\Delta t=\tau_2-\tau_1 \Rightarrow H_0$
- Δt is measured by aligning the shifted light curves of multiple lensed images
- Estimates of H_0 depend on lens model (i.e. on deflection potential ψ)



Measuring the time delays from the light curves of the images

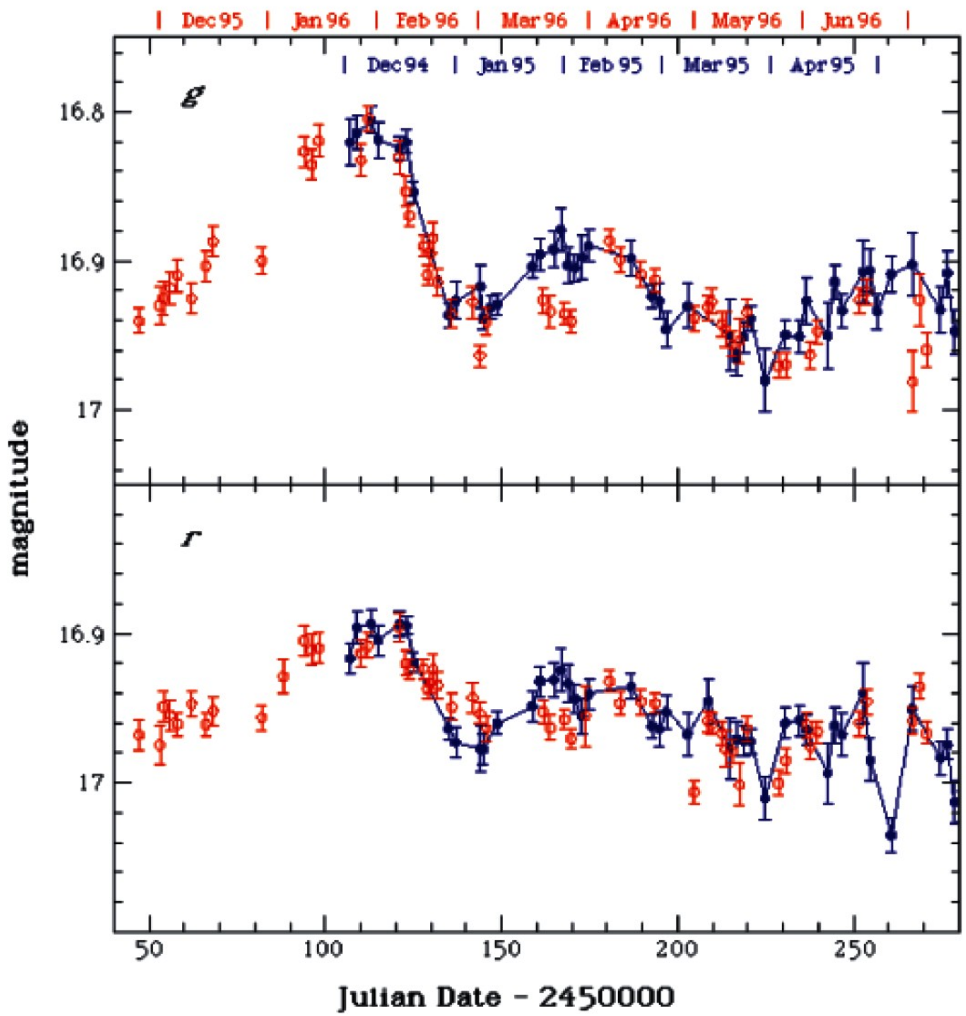
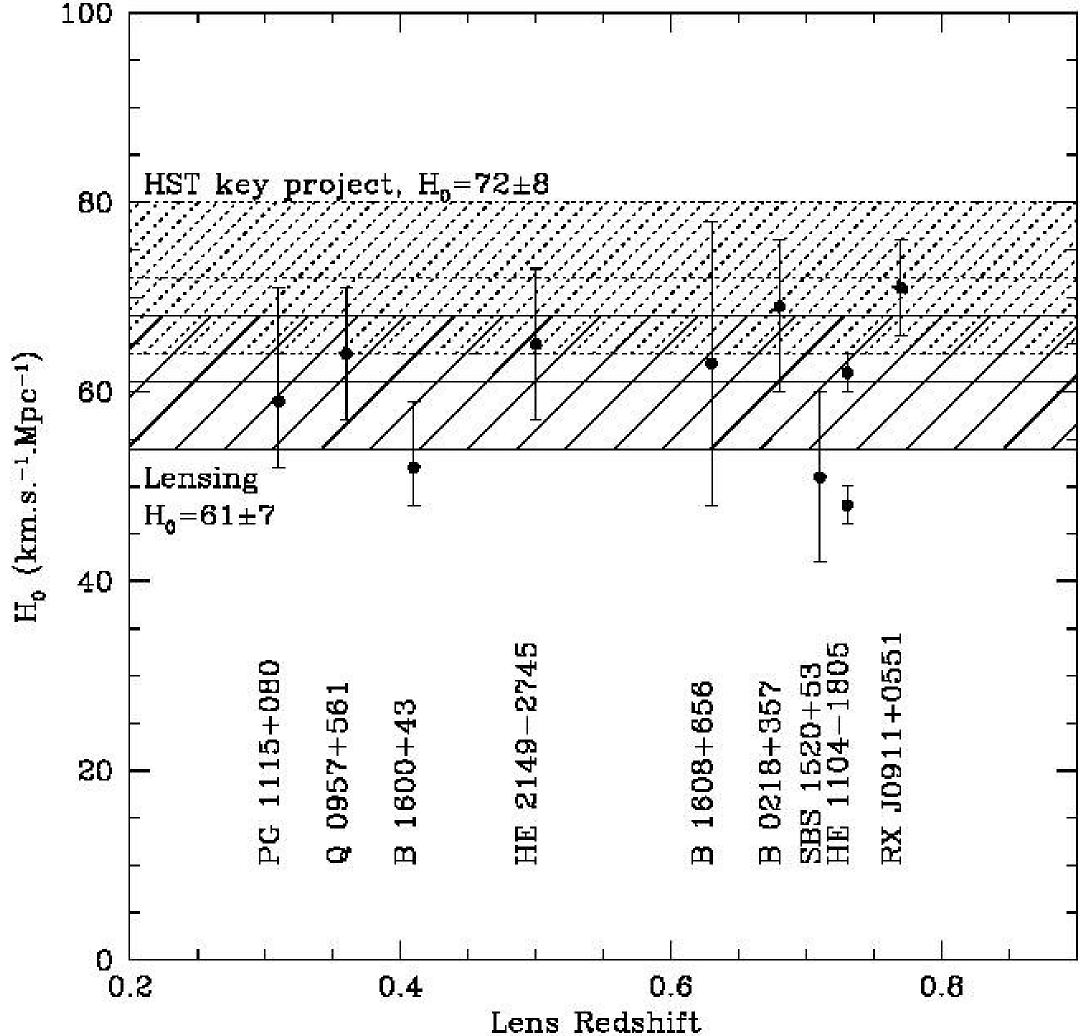


Fig. 9. Light curves of the two images of the QSO 0957+561A,B in two different filters. The two light curves have been shifted in time relative to each other by the measured time delay of 417 days, and in flux according to the flux ratio. The sharp drop measured in image A in Dec. 1994 and subsequently in image B in Feb. 1996 provides an accurate measurement of the time delay (data from Kundić et al. 1997)

Determining H_0 from lensing time delays

Table 1.1. *Time Delay Measurements*

System	N_{im}	Δt (days)
HE1104–1805	2	161 ± 7
PG1115+080	4	25 ± 2
SBS1520+530	2	130 ± 3
B1600+434	2	51 ± 2
HE2149–2745	2	103 ± 12
RXJ0911+0551	4	146 ± 4
Q0957+561	2	417 ± 3
B1608+656	4	77 ± 2
B0218+357	2	10.5 ± 0.2
PKS1830–211	2	26 ± 4
B1422+231	4	(8 ± 3)



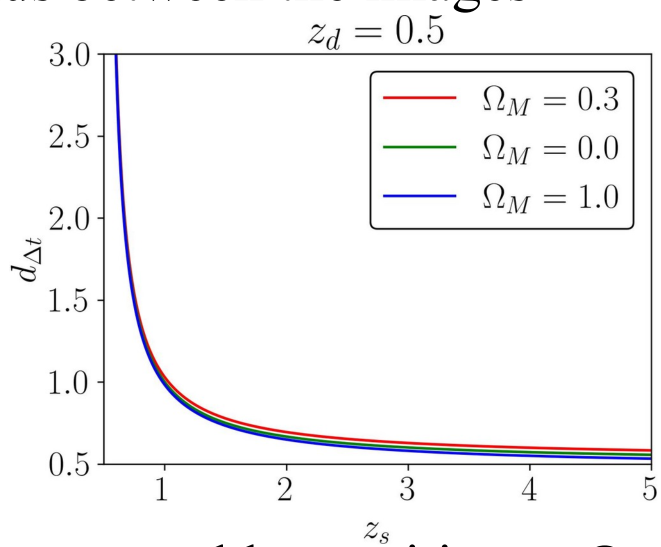
Schechter, 2005, IAU Symposium 225, 281

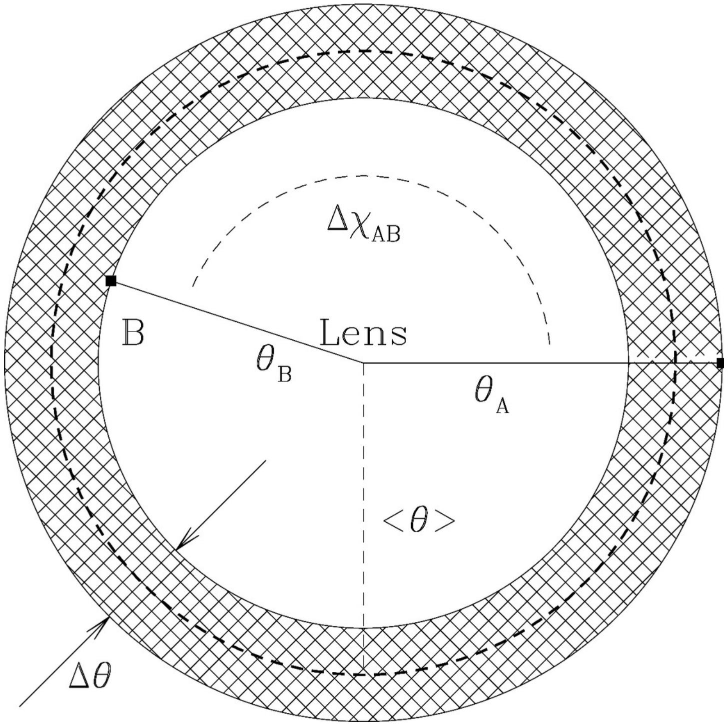
Time delay of the circularly symmetric lenses

- $D_{\Delta t} = (1+z_d)\dfrac{D_d D_s}{D_{ds}} = \dfrac{c}{H_0} d_{\Delta t},$ where $d_{\Delta t} = \dfrac{d_C(z_d)\cdot d_C(z_s)}{d_C(z_s) - d_C(z_d)}$ is **dimensionless time-delay distance**, and $d_C(z) = \displaystyle\int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_\Lambda}}$ is dimensionless comoving distance
- Total time delay between two images is then given by (up to the first order):

$$\boxed{\Delta t \approx \frac{d_{\Delta t}}{H_0}(\theta_A^2 - \theta_B^2)(1 - \langle\kappa\rangle),}$$

where $<\kappa>$ is the mean surface density in the annulus between the images





(Kochanek & Schechter, 2004, astro-ph/0306040)

- $d_{\Delta t}$ is very weakly sensitive to Ω_M and Ω_Λ

Optical depth τ and statistics of strong lenses

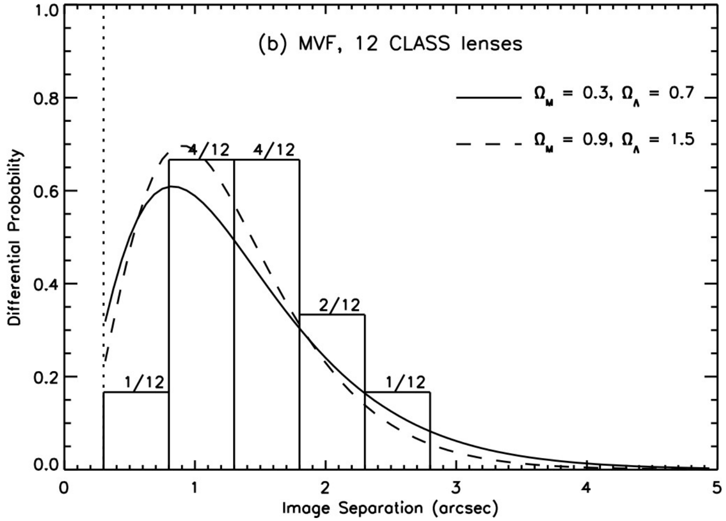
- **τ** - probability of a source being lensed (chance of seeing a lensing event), i.e. probability that at any instant of time a source is within the Einstein ring of a lens
- **Cross section of strong lensing (A)** - area in the lens plane where the separation between the lens and source is sufficiently small for strong lensing to occur: $A = \pi\theta_E^2$
- The total τ is obtained by summing the cross sections of all deflectors between the observer and source, and it depends on cosmological parameters:

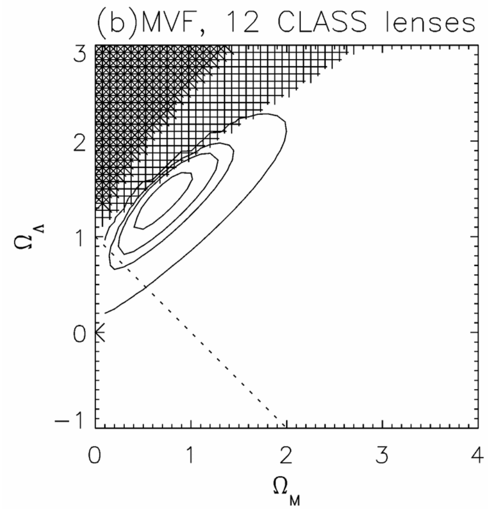
$$\boxed{\tau_{SIS}(z_l, z_s, \Omega_M, \Omega_\Lambda) = \frac{1}{4\pi}\int_0^{z_s} dV \int_0^{\infty} d\sigma \cdot \frac{dn}{d\sigma} \cdot A_{SIS}(\sigma, \Omega_M, \Omega_\Lambda, z_l, z_s),}$$ where

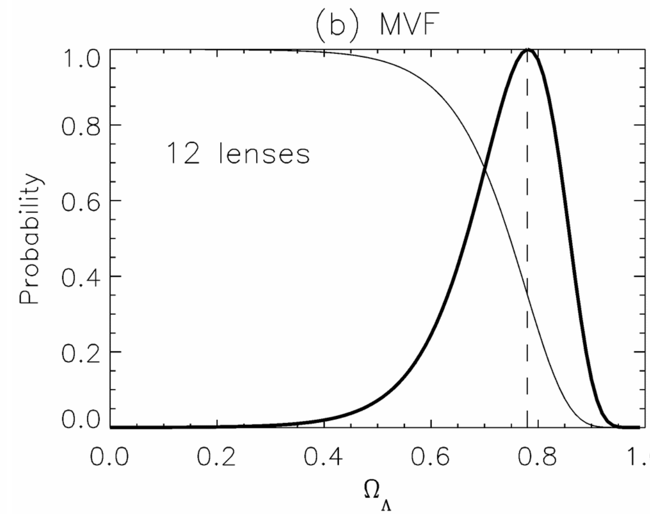
$$V = \frac{4}{3}D_C^3 \Rightarrow dV = 4D_C^2 \cdot dD_C \quad \text{and} \quad A_{SIS} = 16\pi^3 \left(\frac{\sigma}{c}\right)^4 \left(\frac{D_{ls}}{D_s}\right)^2$$

- **Statistical distributions** obtained from differential optical depth $d\tau$: distribution per image separations $\Delta\theta$ $(d\tau/d\Delta\theta)$, distribution per redshift of lens galaxies z_l $(d\tau/dz_l)$ and joint distribution $d^2\tau/(dz_l\, d\Delta\theta)$ per both z_l and $\Delta\theta$
- **Relative probability** of finding a lens at some z_l: $\delta p_l = \frac{d\tau}{dz_l}/\tau$
- Fitting a probability distribution from an observed sample of strong lenses by a modeled prediction $\Rightarrow \Omega_M \wedge \Omega_\Lambda$ (the results do not depend on H_0)
- It is essentially a comoving volume cosmological test

Cosmological parameters from statistics of strong lenses







- Early results (Mitchell et al. 2005, *ApJ*, 622, 81) were not in agreement with with other cosmological tests, while more recent (Cao et al. 2012, *ApJ*, 755, 31; Oguri et al. 2012, *AJ*, 143 120) are

Impact of Gravitational Lensing on Cosmology
Proceedings IAU Symposium No. 225, 2004
Mellier, Y. & Meylan, G. eds.

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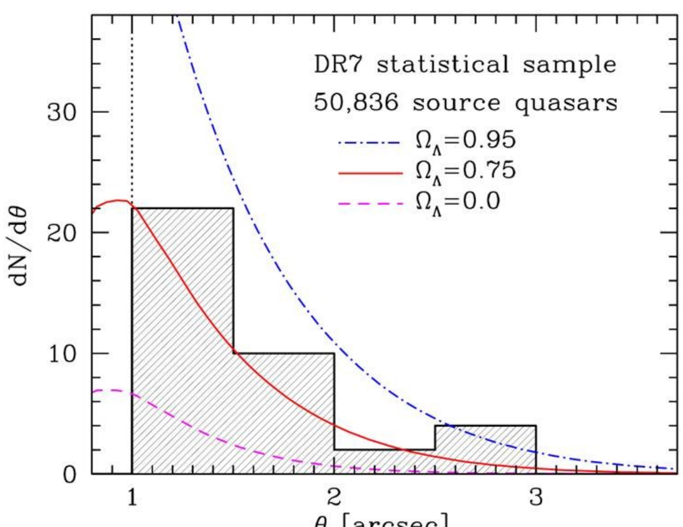
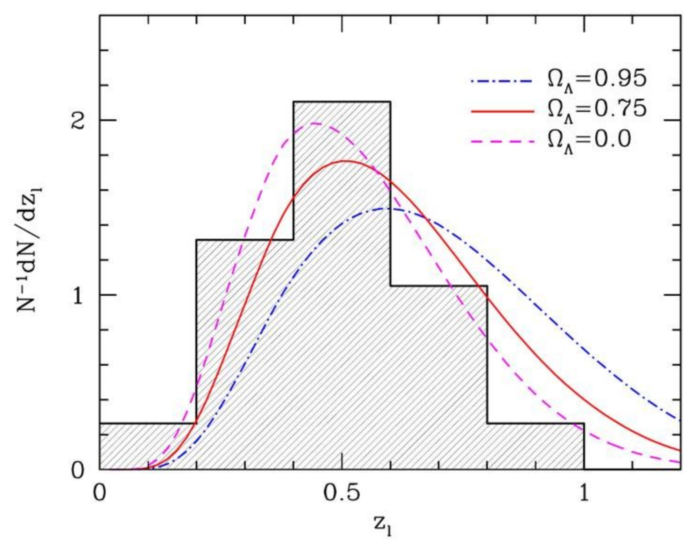
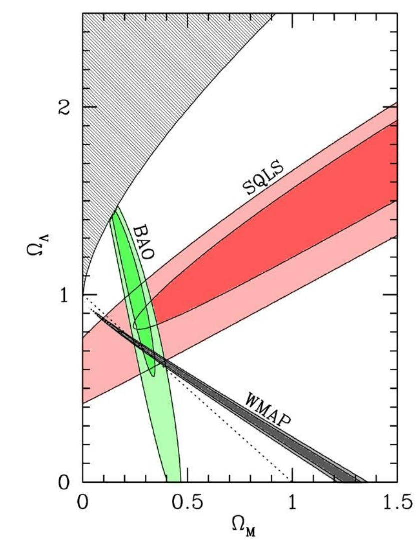
Quasar Lensing Statistics and Ω_Λ: What Went Wrong?

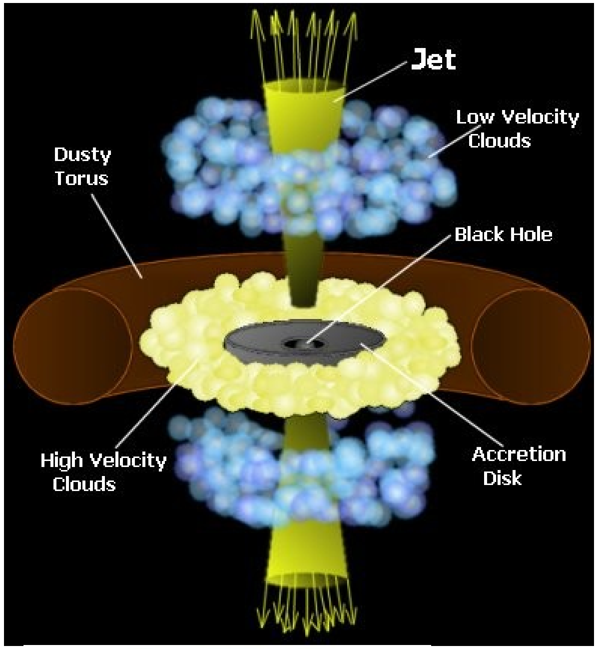
Dan Maoz

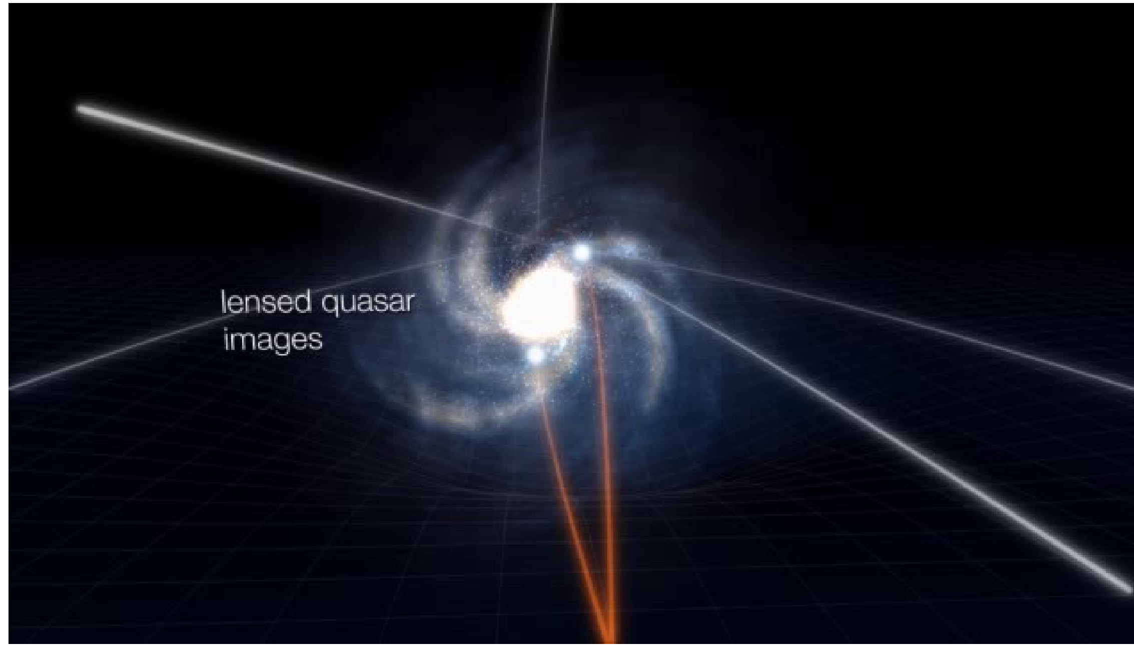


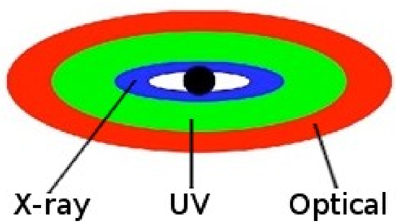




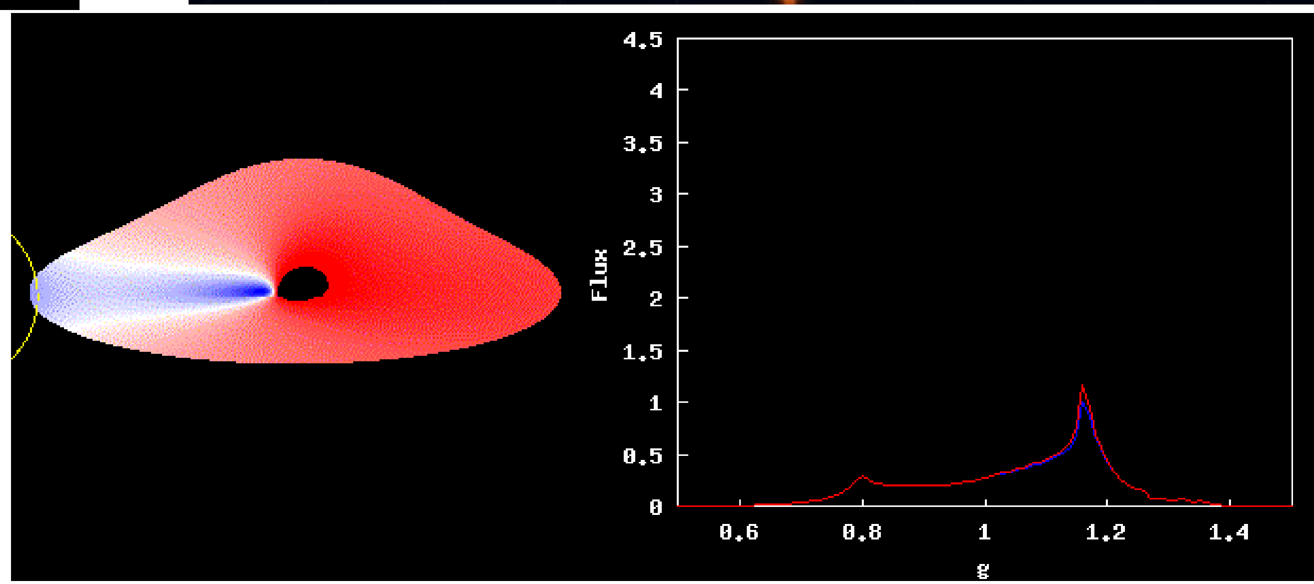
Microlensing applications for studying physics and spacetime geometry in vicinity of SMBHs





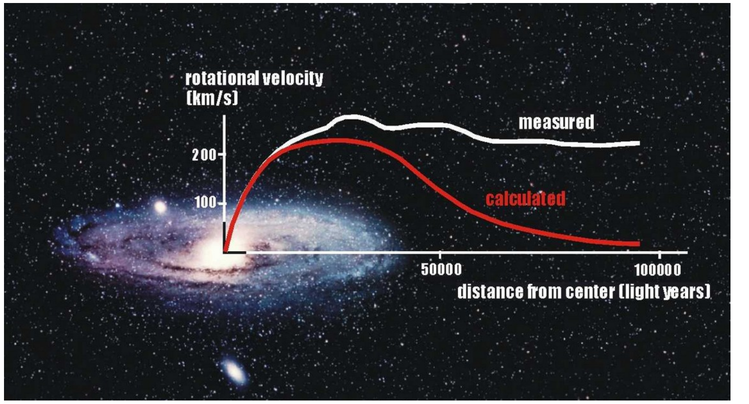


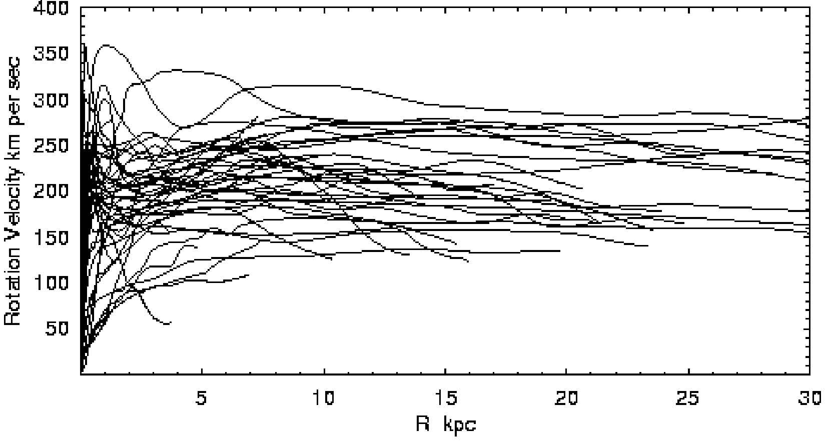
Right: influence of a point-like microlens on X-ray radiation from a relativistic accretion disk around a SMBH



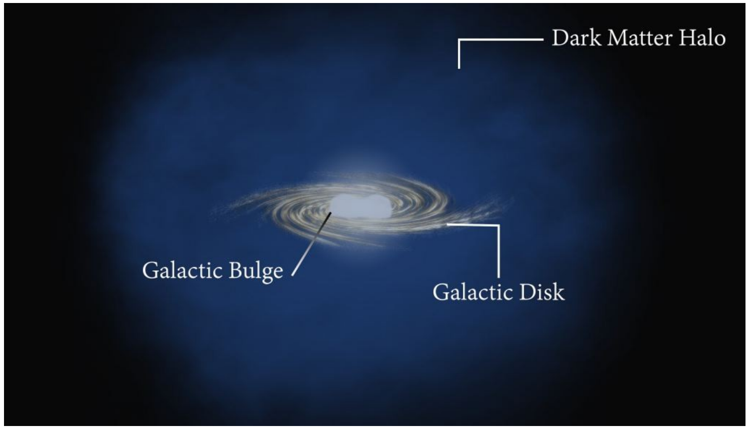
Dark matter

- Zwicky applied virial theorem to the motions of galaxies in the Coma Cluster ⇒ several hundred times more estimated than observable mass ⇒ "dunkle Materie" (Zwicky, 1933, HPA, 6, 110)
- Vera Rubin in the 1960s and 1970s: the observed rotation curves of spiral galaxies are flat ⇒ 6 times as much dark as visible mass
- DM is composed from non-baryonic particles which are so weakly interacting that they move purely under the influence of gravity ⇒ it can be directly detected only by weak lensing

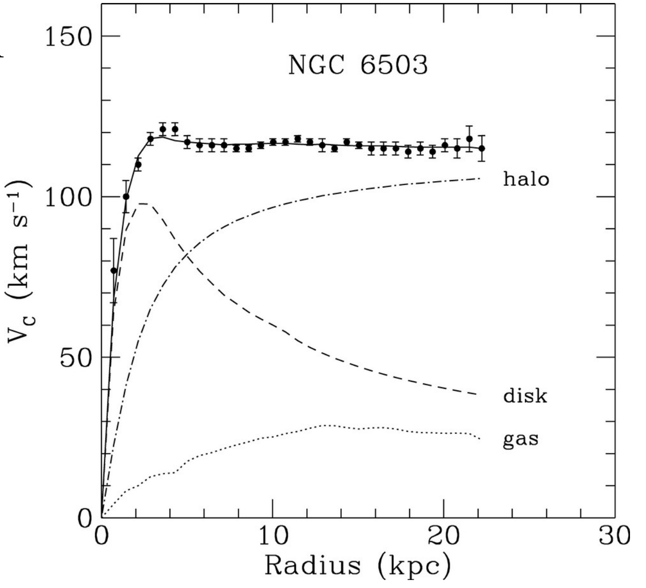




- Hypothesis: a spherical **dark matter halo** around a spiral galaxy (Navarro, Frenk & White, 1996, ApJ, 462, 563):

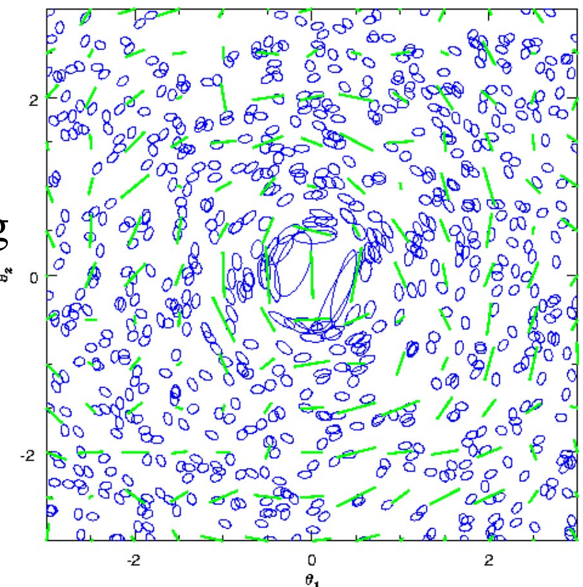


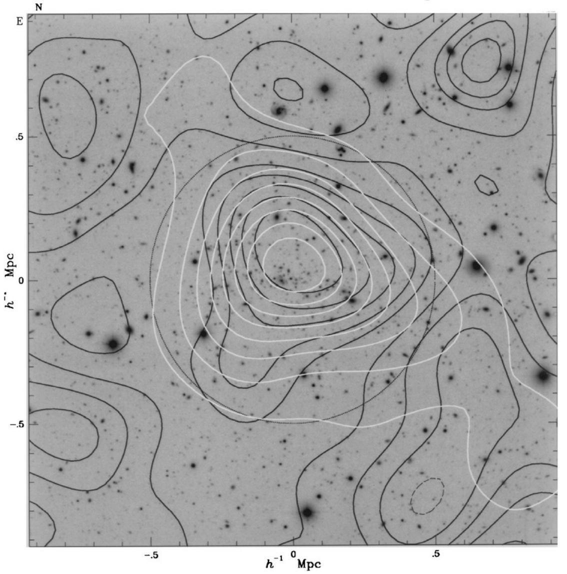
$$\rho(r) = \frac{\rho_0}{\frac{r}{r_0}\left(1+\frac{r}{r_0}\right)^2}$$



Detection of dark matter by weak lensing

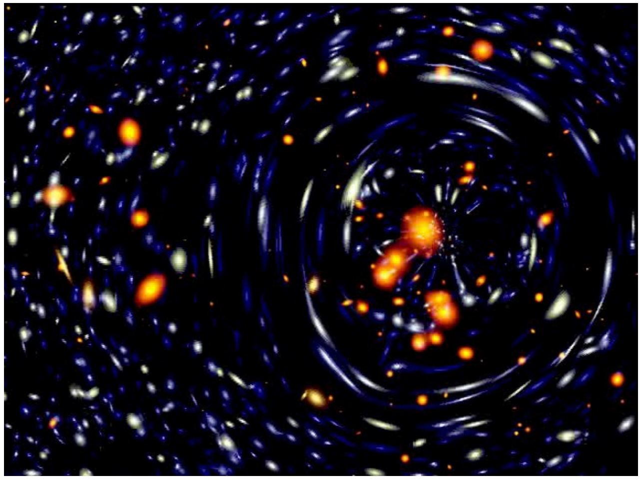
- An observed galaxy ellipticity is a combination of its intrinsic ellipticity and shear γ
- Shear γ can be estimated by averaging over many galaxy images, assuming that the intrinsic ellipticities are *randomly oriented*
- Since both γ and κ are second partial derivatives of the deflection potential ψ, they are linearly related
- **Mass reconstruction:** obtaining the surface mass density κ from the measured values of shear γ

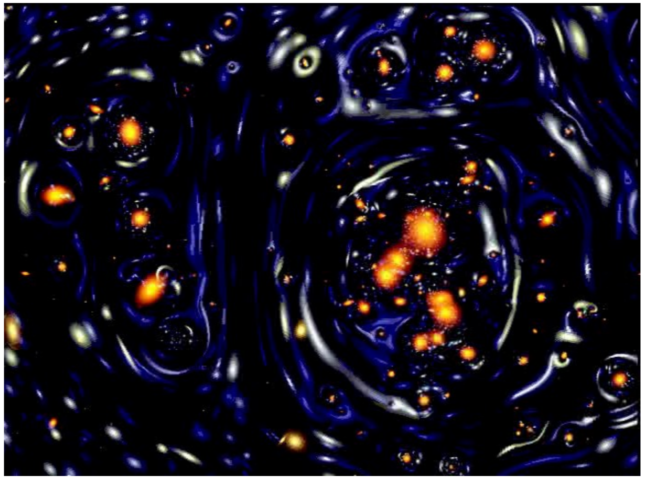




Top right: reconstructed mass distribution (black) in the cluster of galaxies MS1054−03, obtained from ≈2400 measured galaxy ellipticities, and compared to the observed light distribution (white)

Right: gravitational lensing by a massive cluster of galaxies with two different distributions of the same amount of the dark matter over the cluster (orange), causing a particular distortion of the background galaxies (white and blue)





Detection of dark matter in the case of "Bullet Cluster" (1E 0657-558)

THE ASTROPHYSICAL JOURNAL, 648:L109–L113, 2006 September 10

A DIRECT EMPIRICAL PROOF OF THE EXISTENCE OF DARK MATTER[1]

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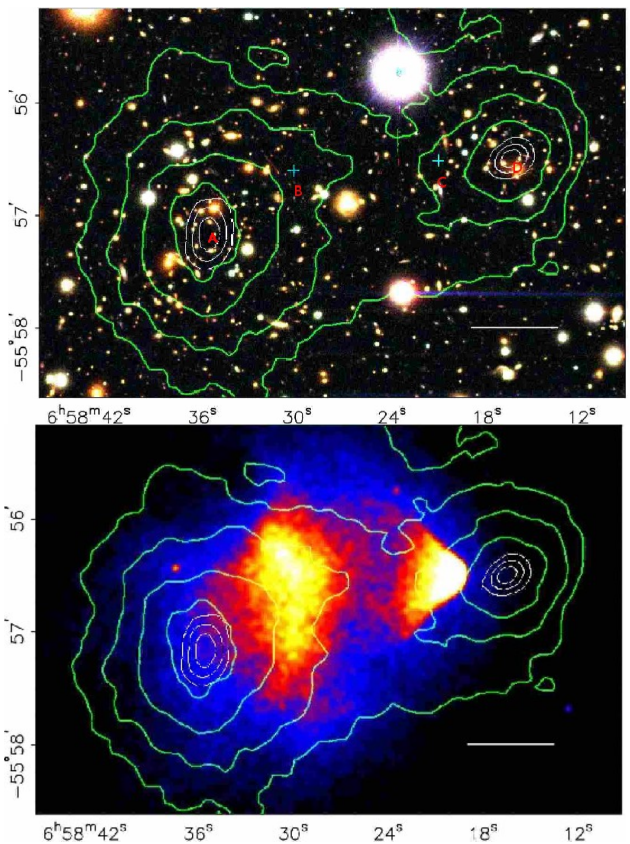
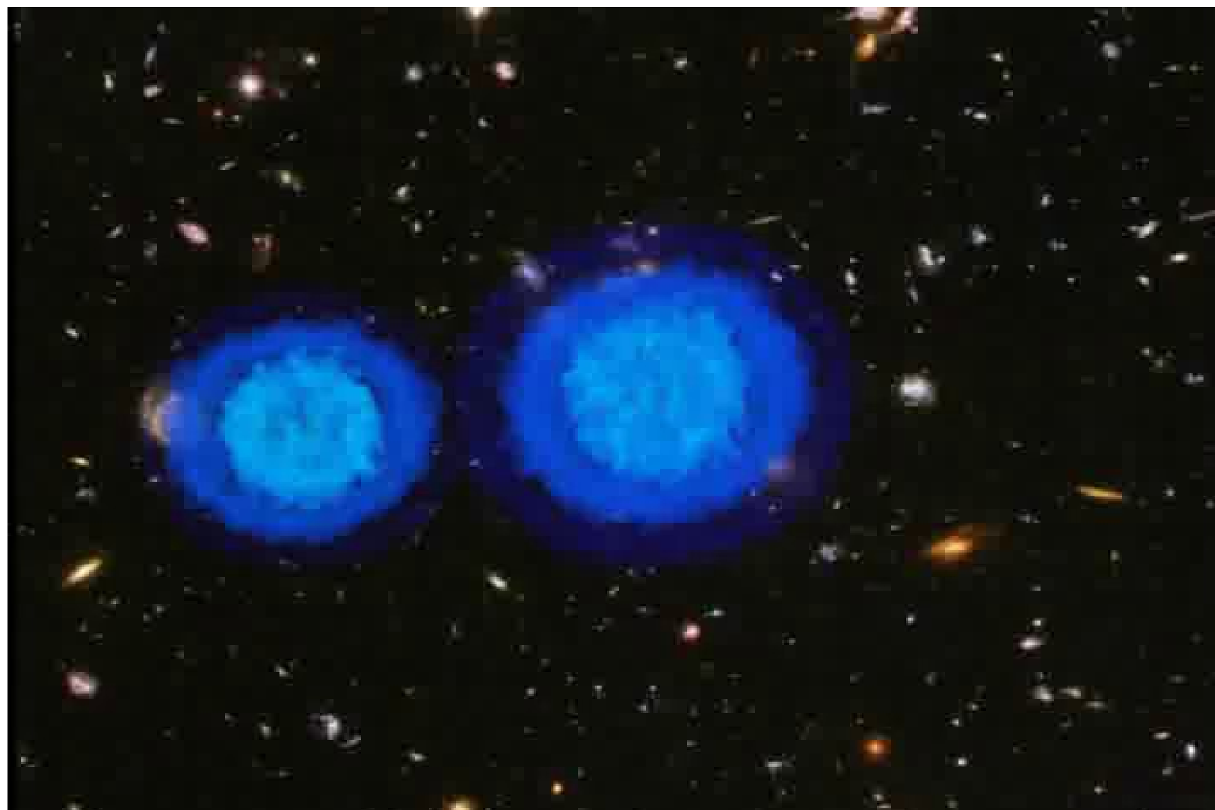
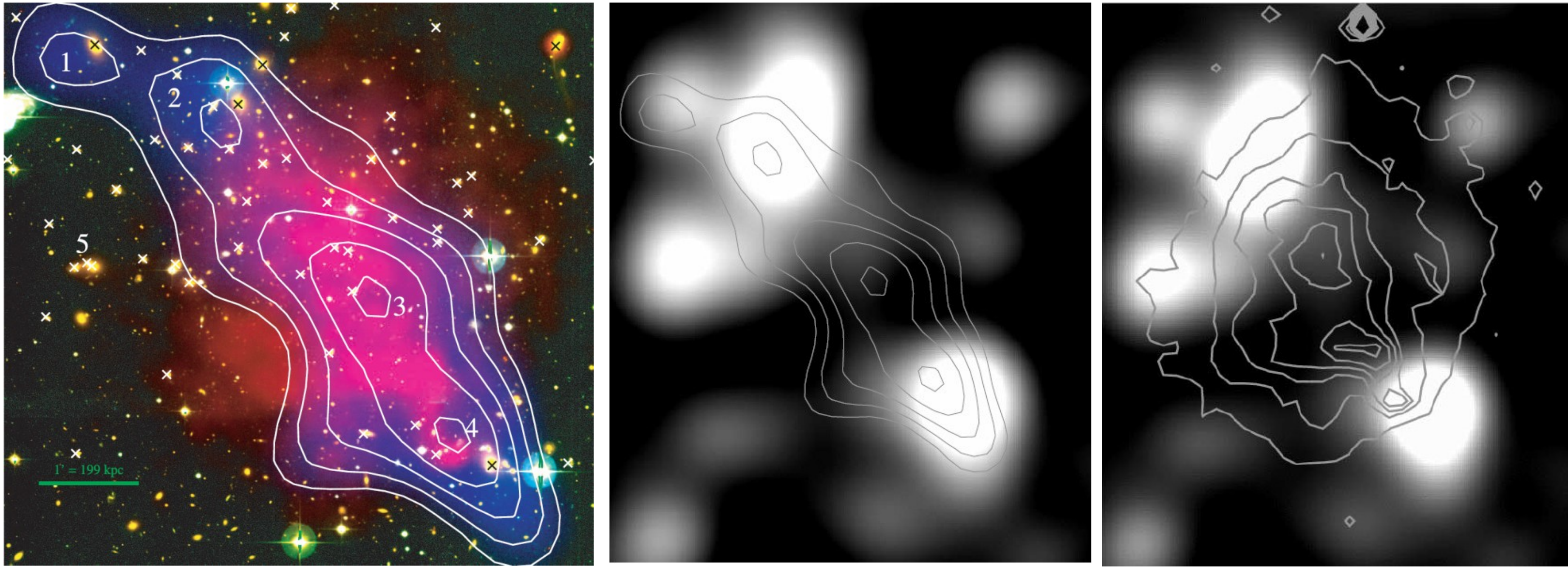
SCOTT W. RANDALL,[5] CHRISTINE JONES,[5] AND DENNIS ZARITSKY[2]
Received 2006 June 6; accepted 2006 August 3; published 2006 August 30





Counterexample: a dark core in Abell 520



THE ASTROPHYSICAL JOURNAL, 668:806–814, 2007 October 20

A DARK CORE IN ABELL 520[1]

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Received 2007 February 10; accepted 2007 June 18

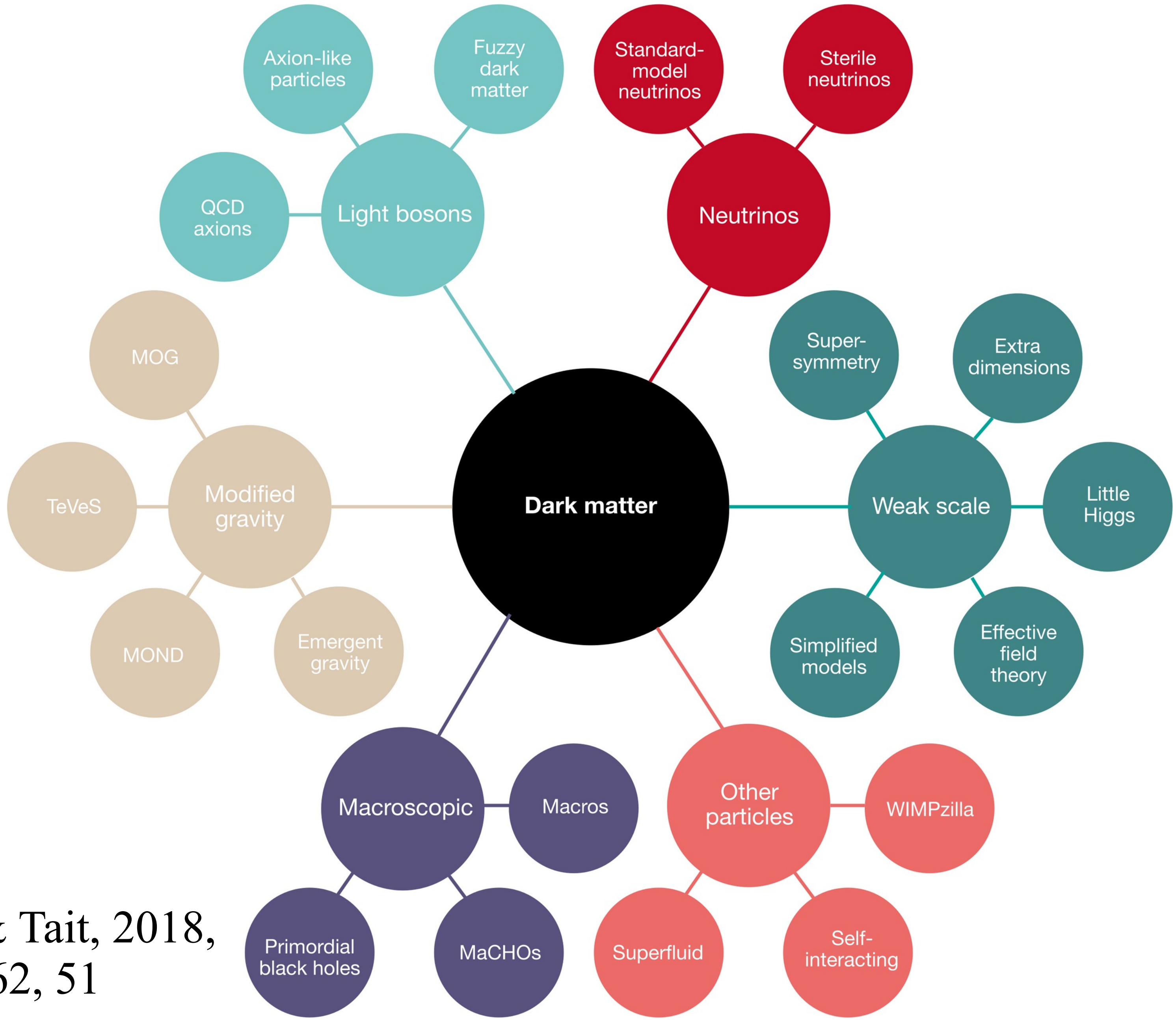
- Bullet cluster: high infall velocities of subclusters around massive main clusters are incompatible with their ΛCDM predictions

THE ASTROPHYSICAL JOURNAL, 718:60–65, 2010 July 20

BULLET CLUSTER: A CHALLENGE TO ΛCDM COSMOLOGY

JOUNGHUN LEE[1] AND EIICHIRO KOMATSU[2]

Possible solutions to the dark matter problem

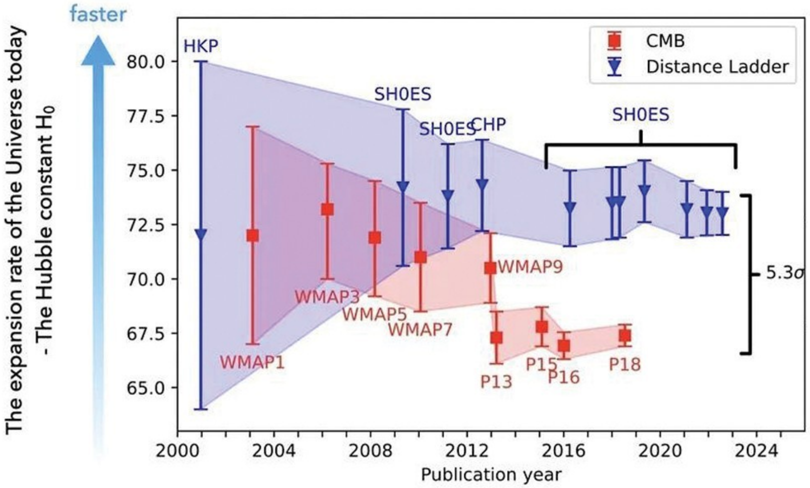


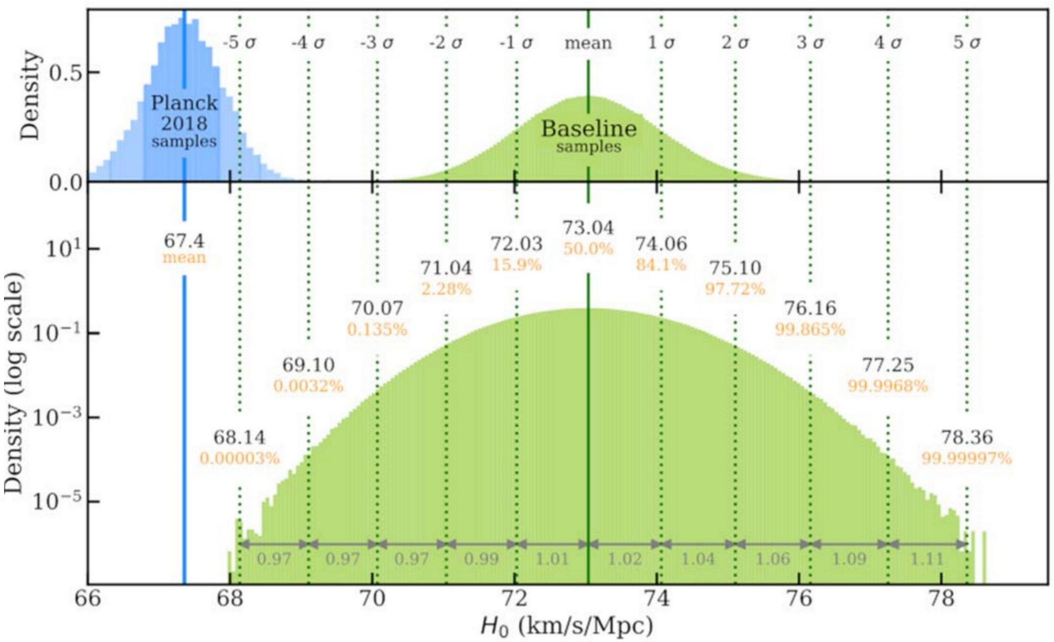
Bertone & Tait, 2018, Nature, 562, 51

Problems with standard ΛCDM model

1. Hubble tension: significant discrepancies between H_0 values obtained from CMBR (early universe) and SN Ia (late universe)

- SN Ia: $H_0 = 74.0 \pm 1.4$ km s^{-1} Mpc^{-1} (Riess et al. 2019, ApJ, 876, 85)
- Planck, 2018 (with 1% precision): $H_0 = 67.4 \pm 0.5$ km s^{-1} Mpc^{-1}
- Discrepancy > 5σ and is higher than systematic errors in the data
- 2σ = curiosity, 3σ = tension, 4σ = discrepancy or problem, 5σ = crisis



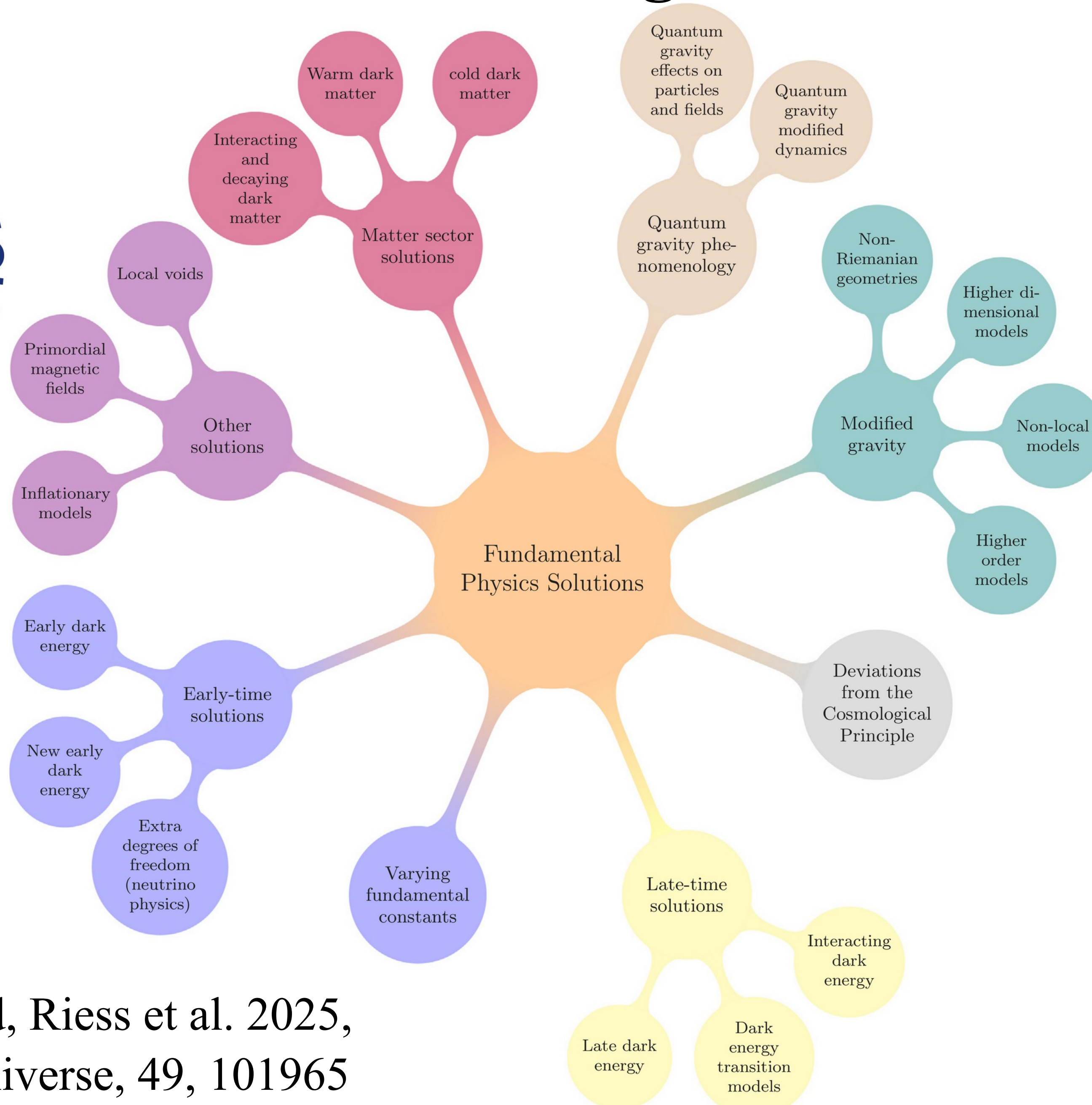


(Right figure from Riess et al. 2022, *ApJL*, 934, L7)

2. Cosmological constant problem (vacuum catastrophe): measured value of Λ is 10^{120} times smaller than the theoretical amount of vacuum energy density predicted by quantum field theory (Weinberg, S. 1989, RvMP, 61, 1)

- The worst theoretical prediction in the history of science

Possible solutions to the cosmological tensions



Di Valentino, Levi Said, Riess et al. 2025,
Physics of the Dark Universe, 49, 101965

Exam question

1. Gravitational lenses and their cosmological applications

Literature

Textbook:

- *Gravitational Lensing*: Strong, Weak and Micro, Book Series: Saas-Fee Advanced Courses
  1. P. Schneider - *Introduction to Gravitational Lensing and Cosmology*
  2. C. S. Kochanek - *Strong Gravitational Lensing*
  3. P. Schneider - *Weak Gravitational Lensing*
  4. J. Wambsganss - *Gravitational Microlensing*

Article:

- Di Valentino et al. 2025, *The CosmoVerse White Paper: Addressing observational tensions in cosmology with systematics and fundamental physics*, Physics of the Dark Universe, 49, 101965

Exercise 1

Estimate the mass of the lensing galaxy of Einstein Cross (Q2237+030) from angular separation of its images. Take this separation and the redshifts from CASTLES Gravitational Lens Data Base at: http://www.cfa.harvard.edu/castles/.

Assume the flat cosmological model with H_0 = 71 km/s/Mpc, Ω_M = 0.27 and Ω_Λ = 0.73, and use the Ned Wright's Javascript Cosmology Calculator to calculate the cosmological distances: http://www.astro.ucla.edu/~wright/CosmoCalc.html

Note that it is convenient to use the gravitational constant expressed in the following units: $G \approx 4.302 \times 10^{-3} \frac{\mathrm{pc}}{M_\odot} \frac{\mathrm{km}^2}{\mathrm{s}^2}$

Compare the obtained mass inside Einstein ring with the corresponding estimates given in Table 1 of Wambsganss & Paczynski, 1994, AJ, 108, 1156.

Exercise 2

Measured time delay between 2 images of gravitational lens system HE2149-2745 is Δt = 103 days, positions of the images in respect to the lens galaxy are: $x_A = 0''.714$, $y_A = -1''.150$, and $x_B = -0''.176$, $y_B = 0''.296$, and redshifts of the source and lens are: $z_s = 2.03$ and $z_d = 0.50$. Estimate the value of H_0, assuming the lens model with $<\kappa> = 0.22$ (Kochanek, 2002, ApJ, 578, 25), as well as the following cosmological model: $\Omega_M = 0$ and $\Omega_\Lambda = 1$.

Solution 1

$z_s = 1.69$, $z_d = 0.04$

Angular separation of the images (size) $\approx 2\theta_E = 1''.78$

$\theta_E = 1''.78 / 2 = 0''.89 = (0.89 / 206265)$ rad $= 4.315 \times 10^{-6}$ rad

Note: 1 rad $= (648000 / \pi)'' \approx 206265''$

$D_d = 161.1$ Mpc

$D_s = 1764.8$ Mpc

$D_{ds} = (D_{Ms}\text{-}D_{Md})/(1+z_s) = (4747.3\text{-}167.5)/(1+1.69)$ Mpc $= 1702.5$ Mpc

$$D = \frac{D_d D_s}{D_{ds}} = 167\,\mathrm{Mpc}$$

Angular Einstein radius $\Rightarrow M = \dfrac{c^2 D\,\theta_E^2}{4G} = 1.6 \times 10^{10} M_\odot$

Table 1 from Wambsganss & Paczynski, 1994, AJ, 108, 1156:

$M \approx 1.5 \times 10^{10} M_\odot$

Solution 2

$$\left.\begin{array}{l}\theta_A = 1''.354 = 6.564\times10^{-6}\text{rad}\\ \theta_B = 0''.344 = 1.668\times10^{-6}\text{rad}\end{array}\right\}\Rightarrow \theta_A^2-\theta_B^2 = 4.03\times10^{-11}$$

$$\Omega_M = 0 \wedge \Omega_\Lambda = 1 \Rightarrow d_C(z) = \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3+\Omega_\Lambda}} = z \Rightarrow$$

$$d_{\Delta t} = \frac{d_C(z_d)\cdot d_C(z_s)}{d_C(z_s)-d_C(z_d)} = \frac{z_d z_s}{z_s - z_d}$$

$$\Delta t \approx \frac{d_{\Delta t}}{H_0}(\theta_A^2-\theta_B^2)\,(1-\langle\kappa\rangle) \Rightarrow \quad H_0 = \frac{z_d z_s(\theta_A^2-\theta_B^2)\,(1-\langle\kappa\rangle)}{(z_s-z_d)\Delta t}$$

$$H_0 = \frac{0.50\cdot 2.03\cdot 4.03\times10^{-11}\cdot 0.78}{(2.03-0.50)\cdot 103\cdot 86400\,\text{s}}\cdot\frac{3.0856776\times10^{19}\,\text{km}}{\text{Mpc}} = 72.3\,\frac{\text{km}}{\text{s}\cdot\text{Mpc}}$$

- **Note**: $1\,\text{Mpc} = 3.0856776\times10^{19}\,\text{km}$