MASS 2023 Course:

Gravitational Lenses

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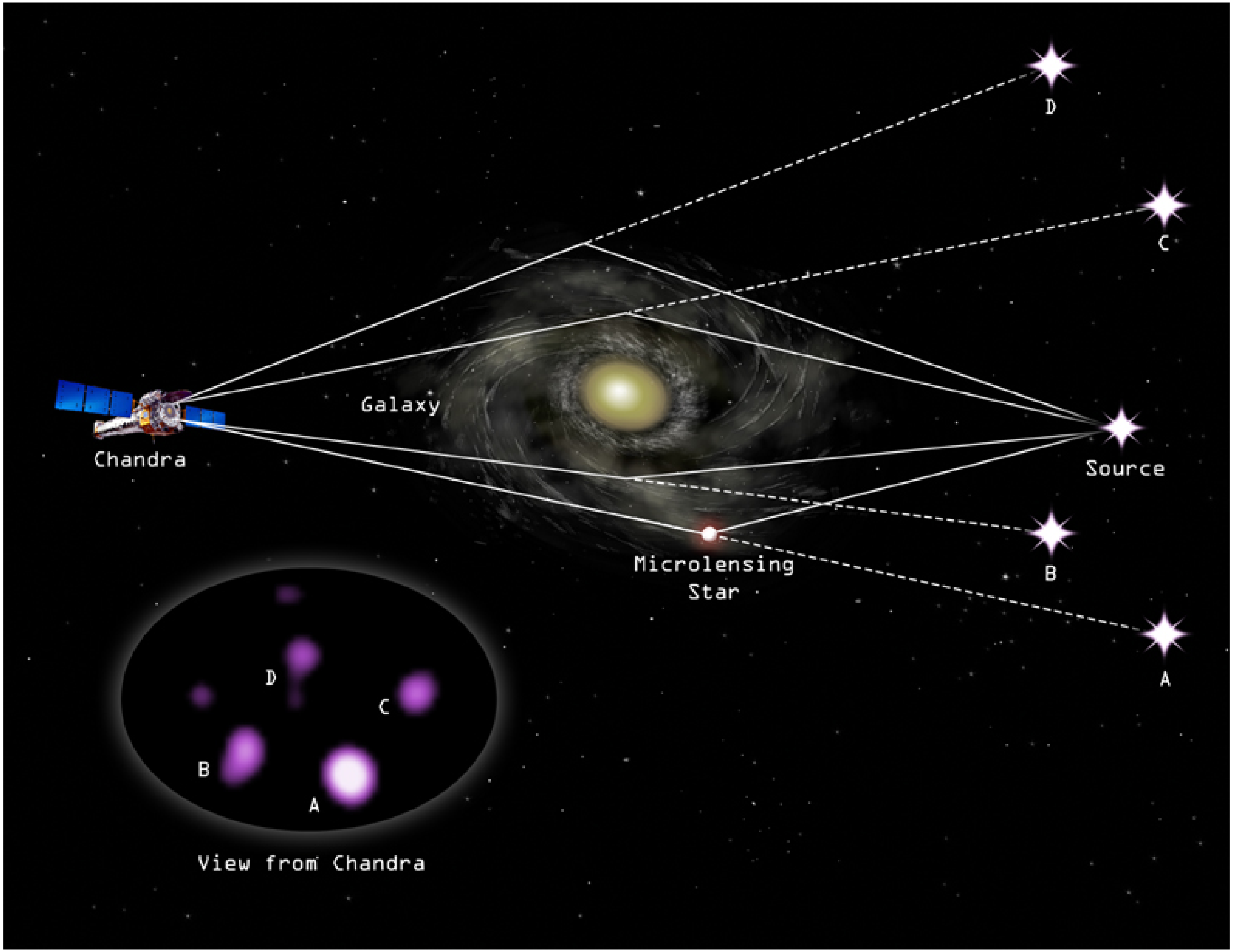
Lecture 05

1. Gravitational lensing theory
   - Geometrically thin lens
   - Deflection angle
   - Lens equation
   - Einstein radius
2. Point-like lens model
   - Solving the lens equation
   - Image positions
   - Magnification of images
3. Exercises

Reminder to Lecture 01



- Light deflection angle: $\alpha = \frac{4GM}{c^2\xi}$

Gravitational lensing theory

- **Geometrically thin lens:** the field equations of GR can be linearized if the gravitational field is weak (i.e. for the small deflection angle), and the ray can be approximated as a straight line near the deflecting mass
- **Lens equation** (see the figure):

$$\vec{\eta} = \frac{D_s}{D_d}\vec{\xi} - D_{ds}\vec{\hat{\alpha}}(\vec{\xi})$$

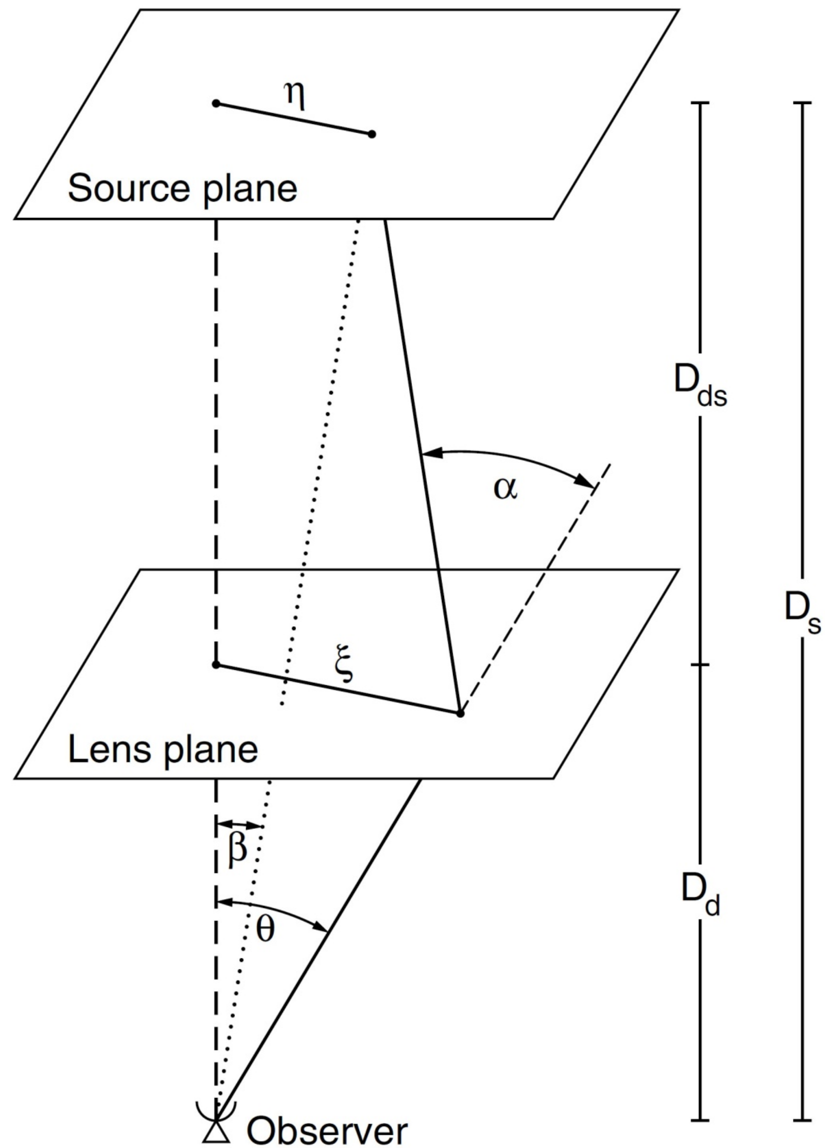
- **Light deflection angle**: $\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4GM}{c^2}\frac{\vec{\xi}}{\xi^2}$
- Angular coordinates $\vec{\beta}$ and $\vec{\theta}$, and **scaled (reduced) deflection angle** $\vec{\alpha}(\vec{\theta})$:

$$\vec{\eta} = D_s\vec{\beta},\ \vec{\xi} = D_d\vec{\theta},\ \vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s}\vec{\hat{\alpha}}(\vec{\theta}) \Rightarrow$$

- Dimensionless lens equation:

$$\boxed{\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})}$$

- Angular diameter distances between the observer and lens, observer and source, and lens and source: $D_d = D_A(0, z_d),\ D_s = D_A(0, z_s),\ D_{ds} = D_A(z_d, z_s)$

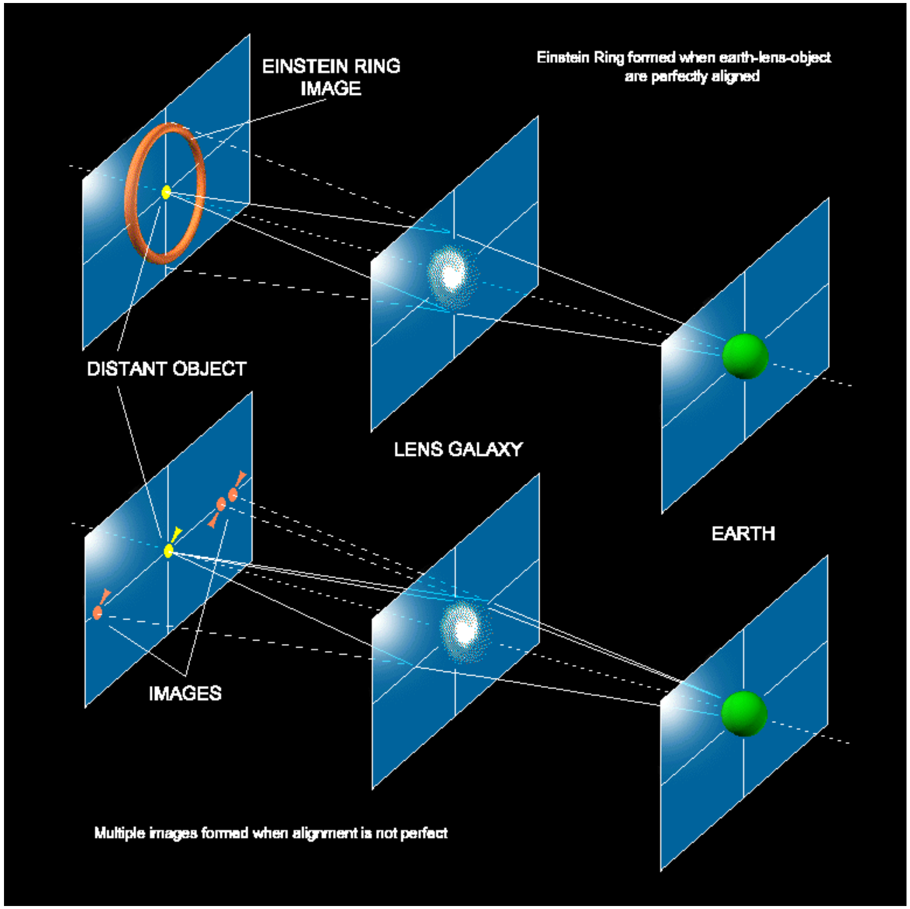


Einstein radius

- Solution of the lens equation for a point mass M and perfect alignment between the observer, lens and source: $\vec{\eta} = \vec{\beta} = 0 \Rightarrow$ **Einstein radius**
- **Linear** (in the lens plane): $\xi_E = \sqrt{\frac{4GM}{c^2}\frac{D_d D_{ds}}{D_s}}$
- **Angular:** $\boxed{\theta_E = \frac{\xi_E}{D_d} = \sqrt{\frac{4GM}{c^2 D}}}$, where

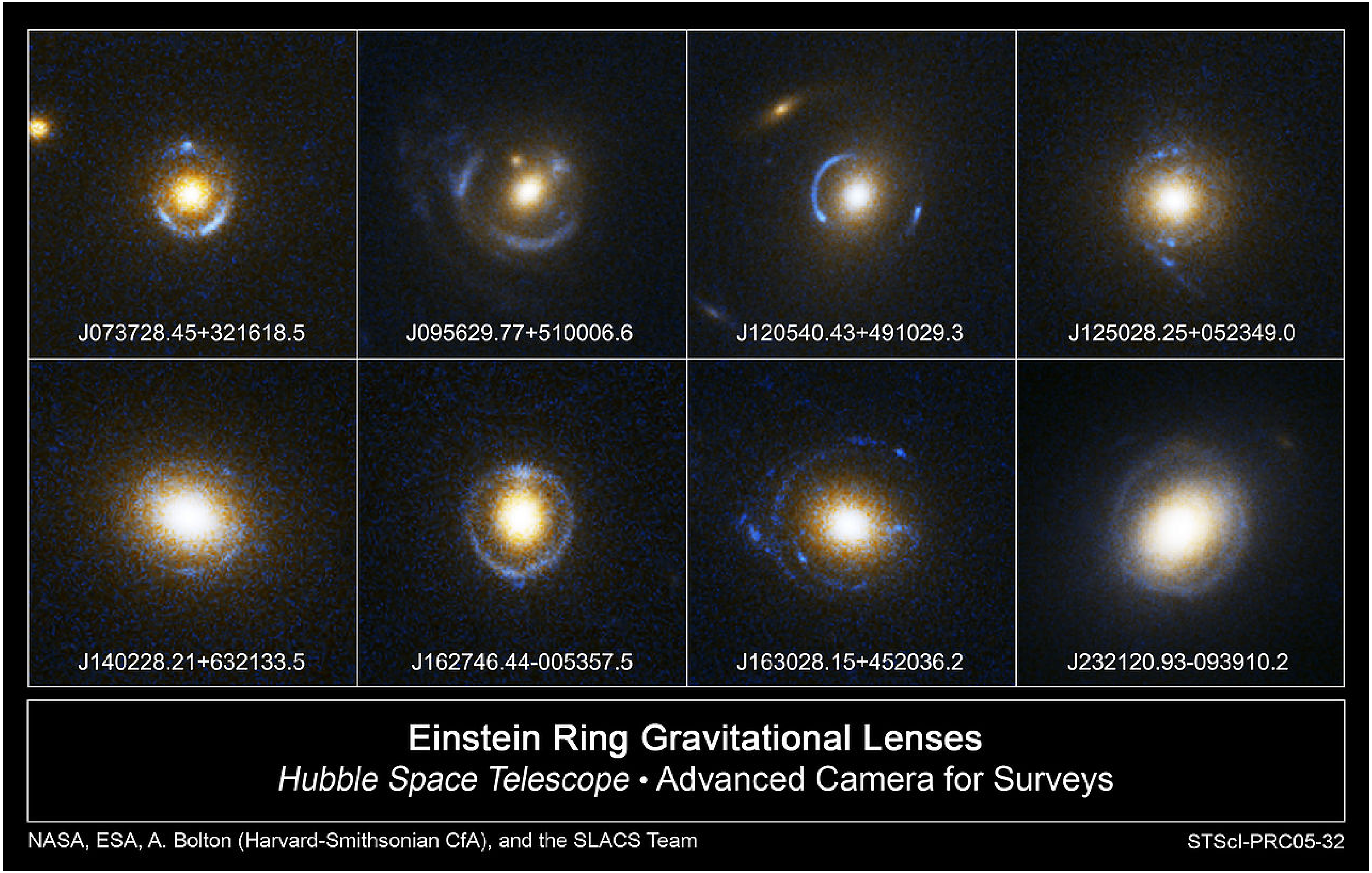
  D is **effective lens distance:** $D = \frac{D_d D_s}{D_{ds}}$
- **Projected** (in the source plane):

  $$\eta_E = \frac{D_s}{D_d}\xi_E = \sqrt{\frac{4GM}{c^2}\frac{D_s D_{ds}}{D_d}}$$
- Typical Einstein radius:
  - for a galaxy: on the order of 1"
  - for galaxy clusters: on the order of 10"
  - for a star: on the order of μas
- Separation between the images is twice the average Einstein radius

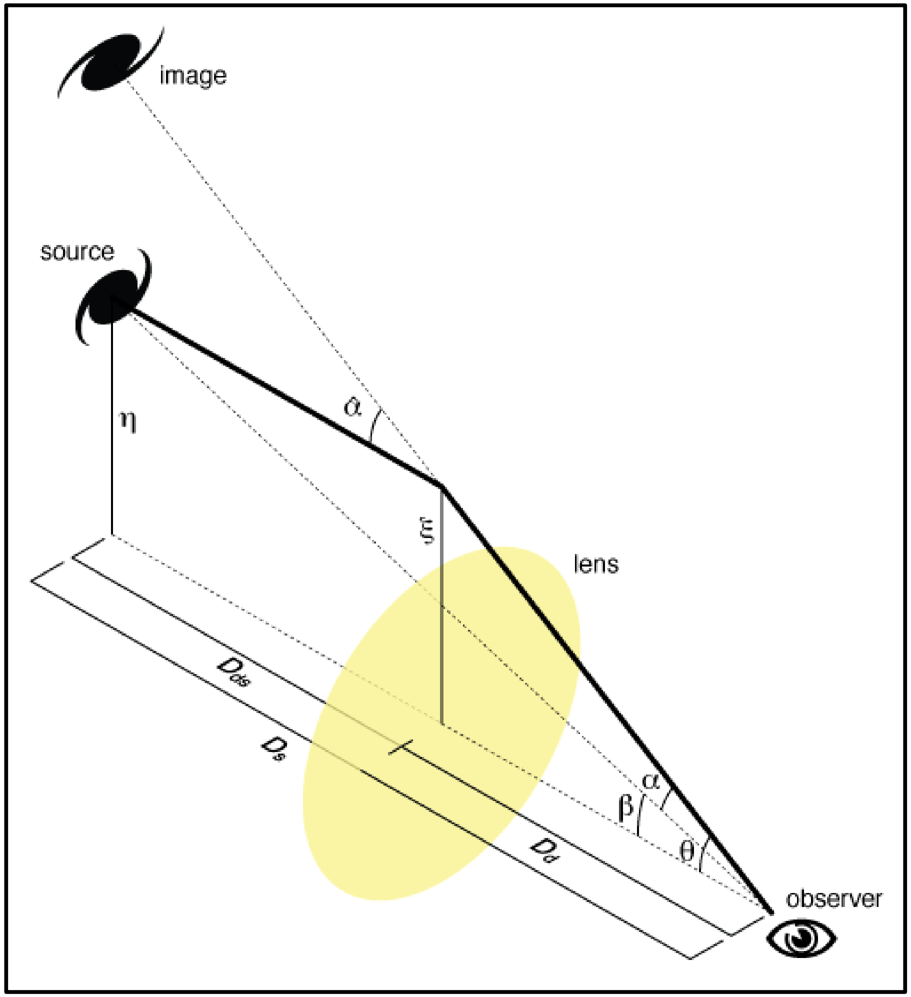


- Powerful method for measuring the masses of distant objects

Observed Einstein rings



Point-like lens model



- **Deflection angle:**

$$\vec{\xi} = D_d \vec{\theta} \Rightarrow \hat{\vec{\alpha}}(\vec{\theta}) = \frac{4GM}{c^2 D_d} \frac{\vec{\theta}}{\theta^2} \Rightarrow$$

$$\vec{\alpha}(\vec{\theta}) = \frac{4GM}{c^2} \frac{D_{ds}}{D_d D_s} \frac{\vec{\theta}}{\theta^2} = \theta_E^2 \frac{\vec{\theta}}{\theta^2}$$

- **Lens equation:**

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}) \Leftrightarrow \boxed{\vec{\beta} = \vec{\theta} - \theta_E^2 \frac{\vec{\theta}}{\theta^2}}$$

- **Image positions:**

(because $\vec{\beta}$ and $\vec{\theta}$ are codirectional)

$$\vec{\beta} = \vec{\theta} - \theta_E^2 \frac{\vec{\theta}}{\theta^2} \Big/ \cdot \vec{\theta} \Rightarrow \theta^2 - \overbrace{\vec{\beta} \cdot \vec{\theta}}^{=\beta\theta} - \theta_E^2 = 0 \Leftrightarrow \theta^2 - \beta\,\theta - \theta_E^2 = 0 \Rightarrow$$

$$\boxed{\theta_{1,2} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)}$$

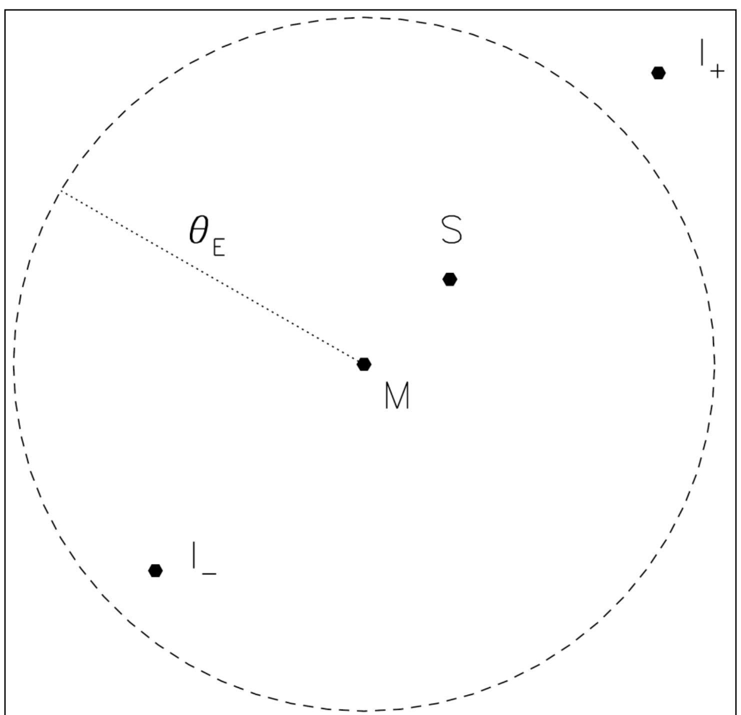
Point-like lens: positions of two images

- Positions of the lens and source normalized to θ_E:

$$\vec{x} = \frac{\vec{\theta}}{\theta_E} \wedge \vec{y} = \frac{\vec{\beta}}{\theta_E} \Rightarrow$$

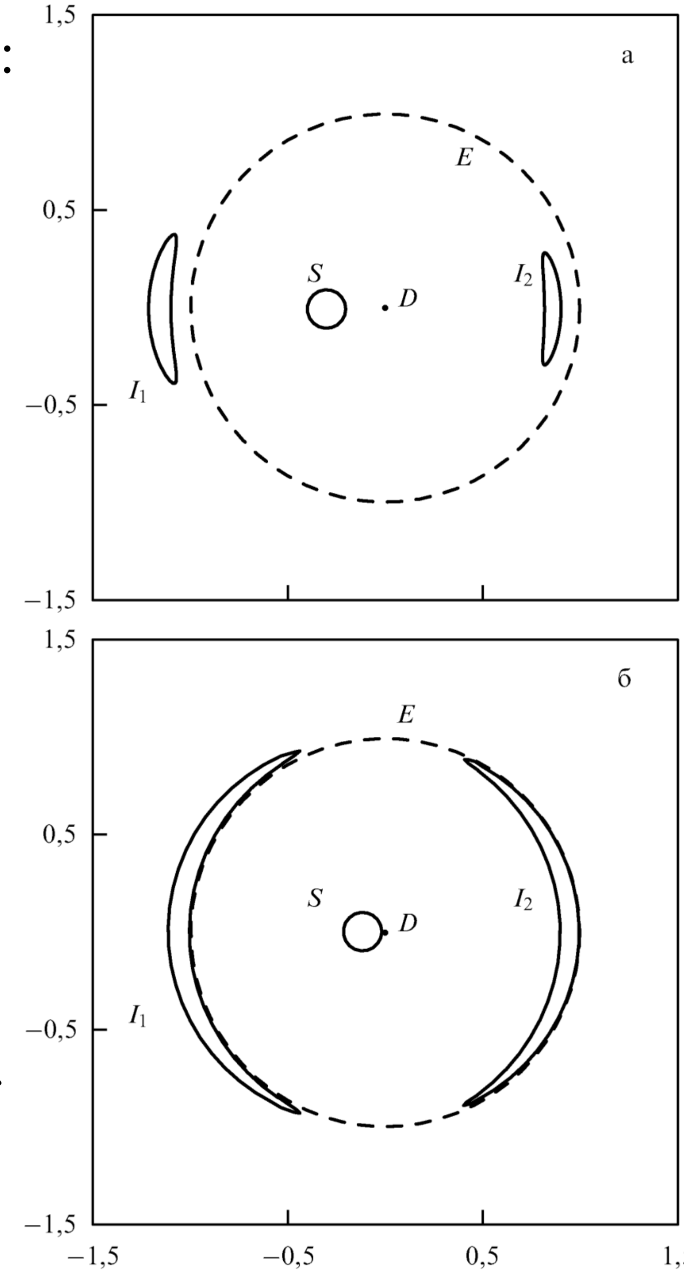
- Lens equation: $\vec{y} = \vec{x} - \frac{\vec{x}}{x^2} \Leftrightarrow \vec{y} = \vec{x} - \frac{\vec{e}}{x} \Rightarrow$

- Image positions: $$\boxed{x_{1,2} = \frac{1}{2}\left(y \pm \sqrt{y^2+4}\right)}$$



Left: images of a point-like source

Right: images of an extended (circular) source



Point-like lens: magnification of images

- Gravitational lensing preserves surface brightness, but changes the apparent solid angle of a source
- **Magnification of an image** - ratio between the solid angles of the image and the source, and for a circularly symmetric lens: $\mu = \frac{\theta}{\beta}\frac{d\theta}{d\beta}$
- In the case of a point lens:

$$\theta^2 - \beta\,\theta - \theta_E^2 = 0 \Rightarrow \beta = \theta - \frac{\theta_E^2}{\theta} \wedge d\beta = \left(1 + \frac{\theta_E^2}{\theta^2}\right) d\theta \Rightarrow$$

$$\boxed{\mu_{1,2} = \left(1 - \left[\frac{\theta_E}{\theta_{1,2}}\right]^4\right)^{-1} \Leftrightarrow \mu_{1,2} = \left(1 - \frac{1}{x_{1,2}^4}\right)^{-1} \Leftrightarrow \mu_{1,2} = \frac{1}{2} \pm \frac{y^2+2}{2y\sqrt{y^2+4}}}$$

- Magnification μ_2 of the image inside the Einstein radius is negative (mirror-inverted): **negative parity**
- $y = 0 \Rightarrow$ Einstein ring of a point source has infinite magnification
- **Total magnification:** sum of the absolute values of two image magnifications: $\mu = |\mu_1| + |\mu_2| = \mu_1 - \mu_2 = \frac{y^2+2}{y\sqrt{y^2+4}} > 1$
- Sum of two image magnifications: $\mu_1 + \mu_2 = 1$

Exam questions

1. Lens equation and Einstein radius
2. Point-like lenses: lens equation, positions and magnification of images

Literature

Textbook:

- *Gravitational Lensing*: Strong, Weak and Micro, Book Series: Saas-Fee Advanced Courses
  1. P. Schneider - *Introduction to Gravitational Lensing and Cosmology*
  2. C. S. Kochanek - *Strong Gravitational Lensing*
  3. P. Schneider - *Weak Gravitational Lensing*
  4. J. Wambsganss - *Gravitational Microlensing*

Exercise 1

Calculate the angular Einstein radius in the case of a lensing galaxy with a mass $M = 10^{12}\, M_\odot$ at a redshift of $z_d = 0.5$ and a source at redshift $z_s = 2.0$. Assume the flat cosmological model with $H_0 = 71$ km/s/Mpc, $\Omega_M = 0.27$ and $\Omega_\Lambda = 0.73$, and use the Ned Wright's Javascript Cosmology Calculator to calculate the cosmological distances: http://www.astro.ucla.edu/~wright/CosmoCalc.html

Note that it is convenient to use the gravitational constant expressed in the following units: $G \approx 4.302 \times 10^{-3} \frac{\mathrm{pc}}{M_\odot} \frac{\mathrm{km}^2}{\mathrm{s}^2}$

Exercise 2

Estimate the mass of the lensing galaxy of Einstein Cross (Q2237+030) from angular separation of its images. Take this separation and the redshifts from CASTLES Gravitational Lens Data Base at: http://www.cfa.harvard.edu/castles/.

Assume the same cosmological model as in previous exercise.

Compare the obtained mass inside Einstein ring with the corresponding estimates given in Table 1 of Wambsganss & Paczynski, 1994, AJ, 108, 1156.

Exercise 3

What is the total magnification in the case of a single point lens, when the source lies on the Einstein radius?

Exercise 4

Consider a point-like lens with Einstein radius of 1" located at coordinate origin, and calculate the images of a point-like background source located at $\vec{\beta} = (0''.3,\ 0''.3)$.

Note that the position vectors of the source $\vec{\beta} = (\beta_x, \beta_y)$ and images $\vec{\theta} = (\theta_x, \theta_y)$ are always codirectional, and thus they have the same unit vectors: $\vec{e} = \frac{\vec{\theta}}{\theta} = \frac{\vec{\beta}}{\beta} \quad \Rightarrow \quad \frac{\theta_{x,y}}{\theta} = \frac{\beta_{x,y}}{\beta}$.

Exercise 5

Consider a point-like lens with Einstein radius of 1" located at coordinate origin, and calculate the images of a circular background source with radius of 0".2, assuming that it is located at the following five different positions along the *x*-axis: -0".5, -0".21, 0.0, 0".21 and 0".5.

Solution 1

$D_d = 1254.5$ Mpc

$D_s = 1748.1$ Mpc

$D_{ds} = (D_{Ms} - D_{Md})/(1+z_s) = (5244.3-1881.7)/(1+2.0)$ Mpc $= 1120.9$ Mpc

$$D = \frac{D_d D_s}{D_{ds}} = 1956.4 \text{ Mpc}$$

Angular Einstein radius: $\theta_E = \sqrt{\frac{4GM}{c^2 D}} \approx 1 \times 10^{-5}\,\text{rad} \approx 2''$

Note: 1 rad = $(648000 / \pi)'' \approx 206265''$

Solution 2

$z_s = 1.69$, $z_d = 0.04$

Angular separation of the images (size) ≈ $2\theta_E = 1''.78$

$\theta_E = 1''.78 / 2 = 0''.89 = (0.89 / 206265)$ rad $= 4.315 \times 10^{-6}$ rad

$D_d = 161.1$ Mpc

$D_s = 1764.8$ Mpc

$D_{ds} = (D_{Ms}-D_{Md})/(1+z_s) = (4747.3-167.5)/(1+1.69)$ Mpc $= 1702.5$ Mpc

$$D = \frac{D_d D_s}{D_{ds}} = 167\,\mathrm{Mpc}$$

$$M = \frac{c^2 D\,\theta_E^2}{4G} = 1.6 \times 10^{10} M_\odot$$

Table 1 from Wambsganss & Paczynski, 1994, AJ, 108, 1156:

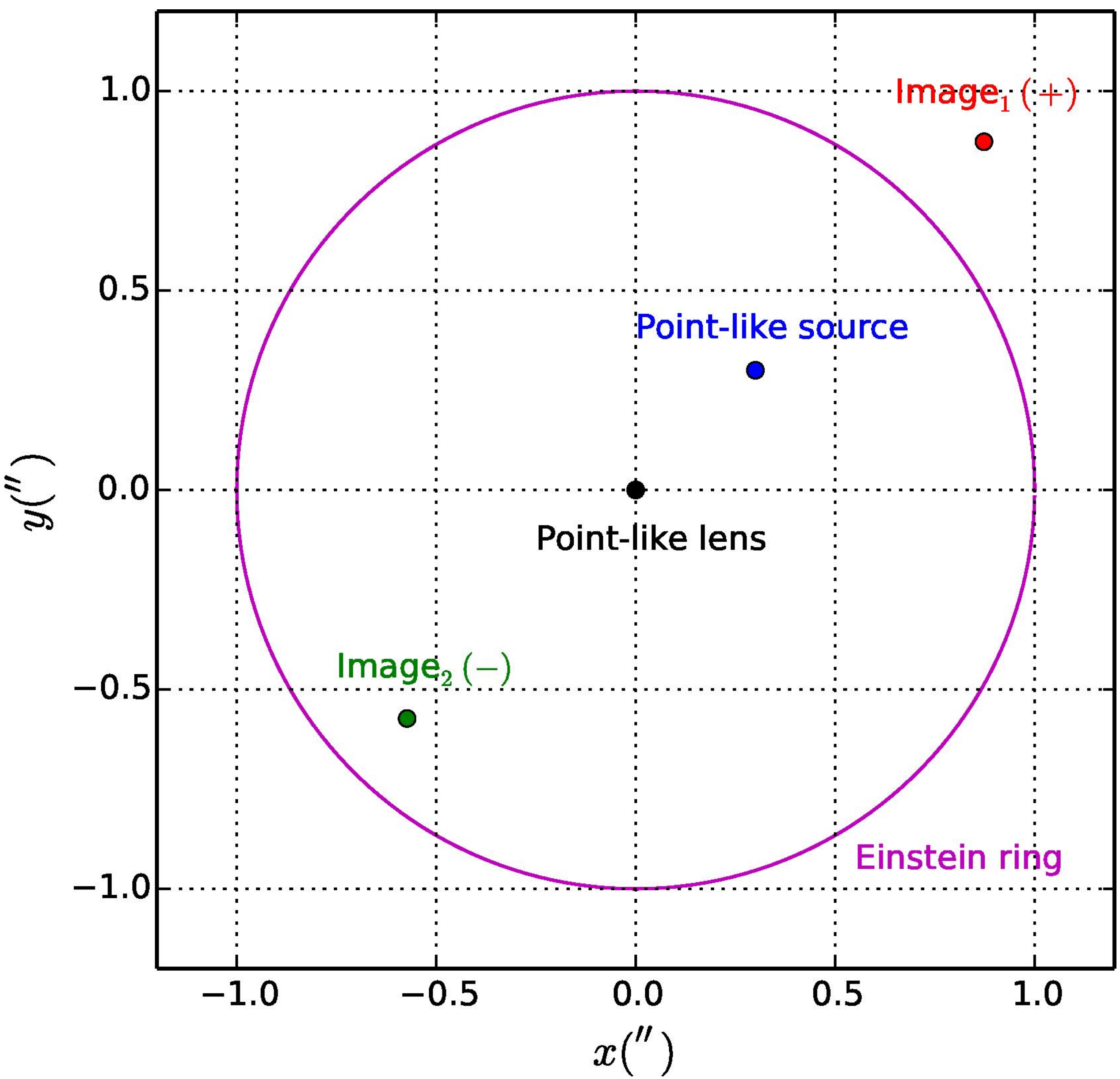
$M \approx 1.5 \times 10^{10}\, M_\odot$

Solution 3

When the source lies on the Einstein radius, we have $\beta = \theta_E$, i.e. $y = 1$, and the total magnification becomes:

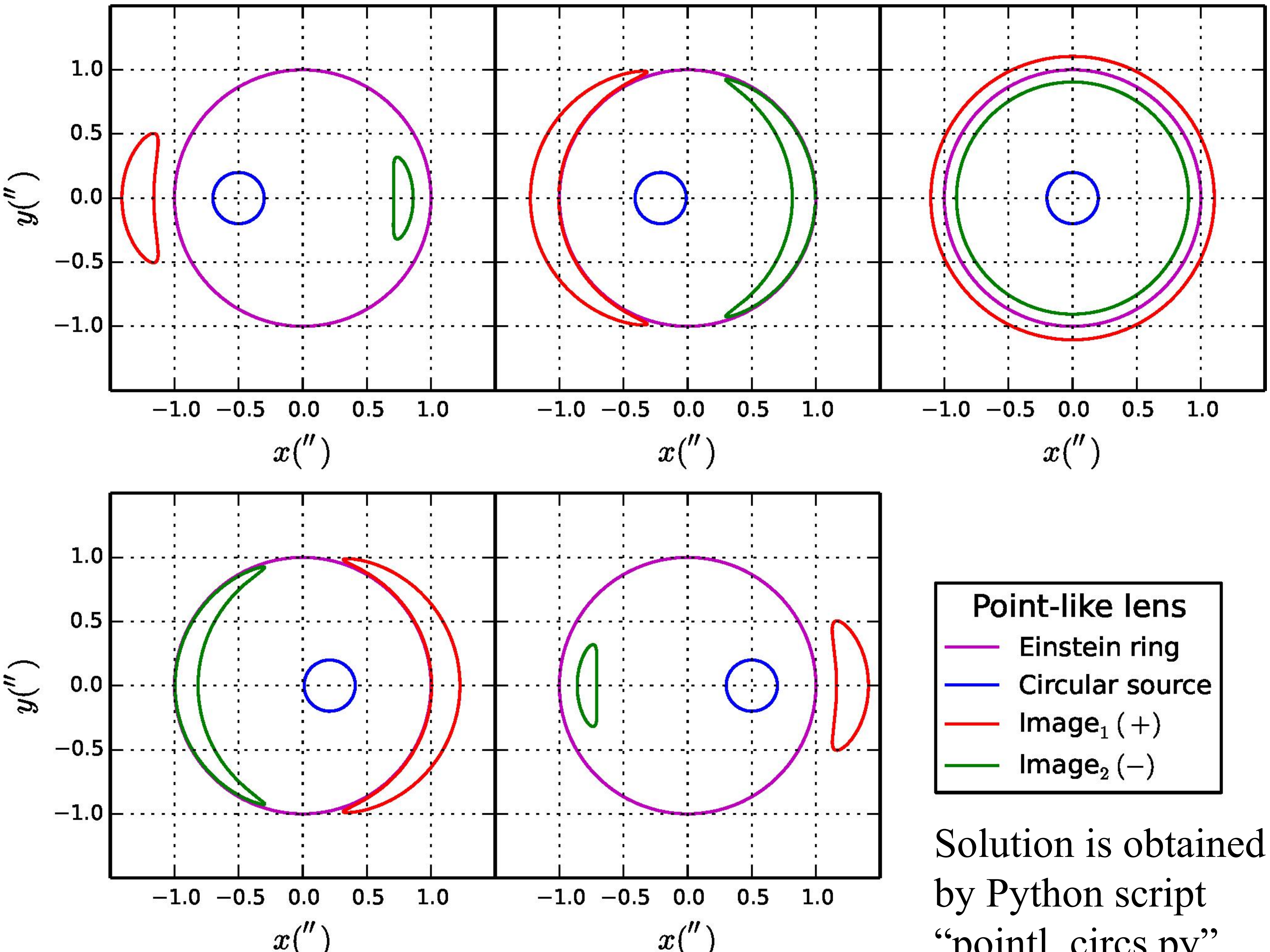
$$\mu = 1.17 + 0.17 = 1.34$$

Solution 4



Solution is obtained by Python script "pointl_points.py"

Solution 5



Solution is obtained by Python script “pointl_circs.py”