3.1 Lensing by a Singular Isothermal Sphere

A simple model for the mass distribution in galaxies assumes that the stars and other mass components behave like particles of an ideal gas, confined by their combined, spherically symmetric gravitational potential. The equation of state of the "particles", henceforth called stars for simplicity, takes the form

$$p = \frac{\rho\, kT}{m}\ , \tag{38}$$

where ρ and m are the mass density and the mass of the stars. In thermal equilibrium, the temperature T is related to the one-dimensional velocity dispersion σ_v of the stars through

$$m\sigma_v^2 = kT\ . \tag{39}$$

The temperature, or equivalently the velocity dispersion, could in general depend on radius r, but it is usually assumed that the stellar gas is isothermal, so that σ_v is constant across the galaxy. The equation of hydrostatic equilibrium then gives

$$\frac{p'}{\rho} = -\frac{GM(r)}{r^2}\ , \quad M'(r) = 4\pi\, r^2\, \rho\ , \tag{40}$$

where $M(r)$ is the mass interior to radius r, and primes denote derivatives with respect to r. A particularly simple solution of eqs. (38) through (40) is

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\,\frac{1}{r^2}\ . \tag{41}$$

This mass distribution is called the *singular isothermal sphere*. Since $\rho \propto r^{-2}$, the mass $M(r)$ increases $\propto r$, and therefore the rotational velocity of test particles in circular orbits in the gravitational potential is

$$v_{\rm rot}^2(r) = \frac{G\, M(r)}{r} = 2\,\sigma_v^2 = \text{constant}\ . \tag{42}$$

The flat rotation curves of galaxies are naturally reproduced by this model.

Upon projecting along the line-of-sight, we obtain the surface mass density

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G}\,\frac{1}{\xi}\ , \tag{43}$$

where ξ is the distance from the center of the two-dimensional profile. Referring to eq. (11), we immediately obtain the deflection angle

$$\hat\alpha = 4\pi\,\frac{\sigma_v^2}{c^2} = (1''\!.4)\,\left(\frac{\sigma_v}{220\,{\rm km\,s^{-1}}}\right)^2\ , \tag{44}$$

which is independent of ξ and points toward the center of the lens. The Einstein radius of the singular isothermal sphere follows from eq. (20),

$$\theta_{\rm E} = 4\pi\,\frac{\sigma_v^2}{c^2}\,\frac{D_{\rm ds}}{D_{\rm s}} = \hat\alpha\,\frac{D_{\rm ds}}{D_{\rm s}} = \alpha\ . \tag{45}$$

Due to circular symmetry, the lens equation is essentially one-dimensional. Multiple images are obtained only if the source lies inside the Einstein ring, i.e. if $\beta < \theta_{\rm E}$. When this condition is satisfied, the lens equation has the two solutions

$$\theta_\pm = \beta \pm \theta_{\rm E}\ . \tag{46}$$

The images at $\theta_\pm$, the source, and the lens all lie on a straight line. Technically, a third image with zero flux is located at $\theta = 0$. This third image acquires a finite flux if the singularity at the center of the lens is replaced by a core region with a finite density.

The magnifications of the two images follow from eq. (26),

$$\mu_\pm = \frac{\theta_\pm}{\beta} = 1 \pm \frac{\theta_{\rm E}}{\beta} = \left(1 \mp \frac{\theta_{\rm E}}{\theta_\pm}\right)^{-1}\ . \tag{47}$$

If the source lies outside the Einstein ring, i.e. if $\beta > \theta_{\rm E}$, there is only one image at $\theta = \theta_+ = \beta + \theta_{\rm E}$.