MASS 2023 Course:

Gravitational Lenses

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Lecture 07

1. Extended lenses:
    - surface mass density
    - convergence
    - circularly symmetric lens
2. Deflection (lensing) potential
3. Simple lens models:
    - Singular Isothermal Sphere (SIS)
    - Softened Isothermal Sphere
    - Isothermal Ellipsoid
4. Exercise

Surface mass density

- 3D mass density $\rho(\vec{r})$ of an **extended lens** can be projected along the line of sight onto the lens plane to obtain the two-dimensional **surface mass density** distribution:

$$\Sigma(\vec{\xi}) = \int_0^{D_s} \rho(\vec{r})dz,$$

where $\vec{r}$ is a three-dimensional vector in space, and $\vec{\xi}$ is a two-dimensional vector in the lens plane

- the two-dimensional **deflection angle** is then the sum of the deflections due to all the mass elements in the plane**:**

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2} d^2\xi'$$

- the **critical surface mass density** is given by the lens mass M "smeared out" over the area of the Einstein ring:

$$\Sigma_{cr} = \frac{M}{\pi\xi_E^2} = \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}} = 0.35\,\mathrm{g\,cm}^{-2} \left(\frac{D}{1\,\mathrm{Gpc}}\right)^{-1}$$

Convergence

- **dimensionless surface mass density** or **convergence** $\boldsymbol{\kappa}$**:**

$$\kappa(\vec{\theta}) := \frac{\Sigma(D_d\vec{\theta})}{\Sigma_{cr}}$$

- a mass distribution which has $\kappa \geq 1$, i.e. $\Sigma \geq \Sigma_{cr}$ produces multiple images for some source positions

- Σ_{cr} is a characteristic value for the surface mass density which separates "weak" from "strong" lenses

Strong vs weak lensing

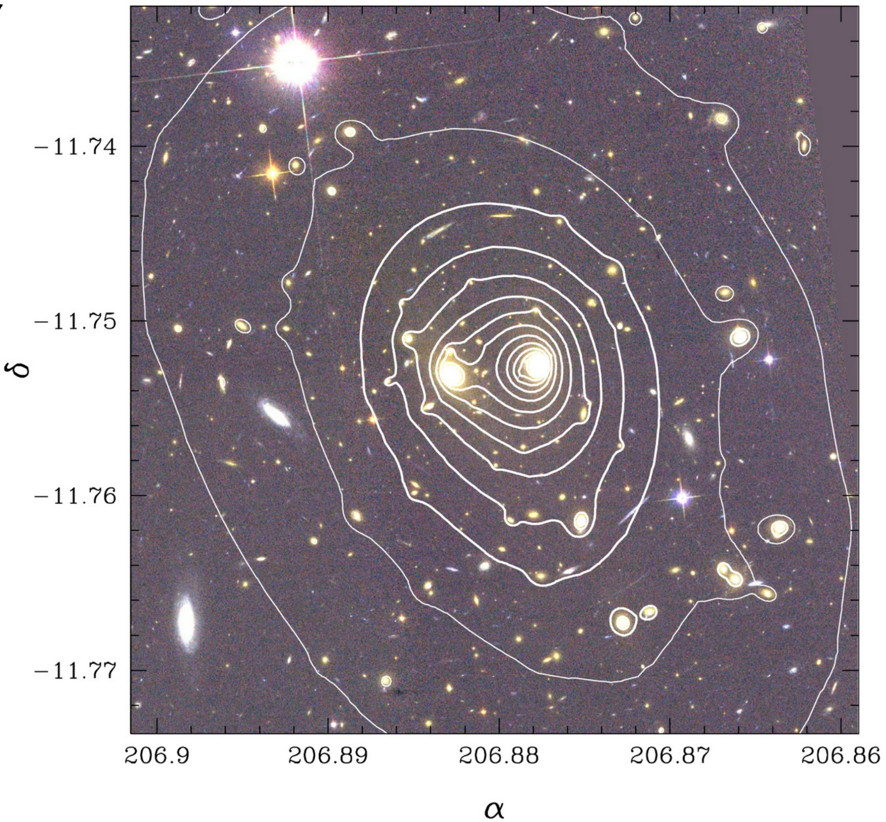


Fig. 4. The average scaled surface mass density contours for a source at redshift 2. The thick contours show the κ levels greater or equal to one, and the thin lines for κ less than one (the contours are separated by $\Delta\kappa = 0.25$, first thick contour level is at $\kappa = 1$).

Deflection potential

- scaled deflection angle expressed in terms of convergence:

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2} d^2\xi' \;\Leftrightarrow\; \vec{\alpha}(\vec{\theta}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \kappa(\vec{\theta'}) \frac{\vec{\theta} - \vec{\theta'}}{|\vec{\theta} - \vec{\theta'}|^2} d^2\theta'$$

- deflection angle as a gradient of the **deflection potential** ***ψ***:

$$\psi(\vec{\theta}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \kappa(\vec{\theta'}) \ln|\vec{\theta} - \vec{\theta'}| d^2\theta' \;\wedge\; \nabla \ln|\vec{\theta}| = \frac{\vec{\theta}}{|\vec{\theta}|^2} \;\Rightarrow\; \boxed{\vec{\alpha} = \vec{\nabla}\psi(\vec{\theta})}$$

- **lens equation**: $\boxed{\vec{\beta} = \vec{\theta} - \vec{\nabla}\psi(\vec{\theta})}$

- relationship between the deflection and Newtonian potential of the lens:

$$\psi(\vec{\theta}) = \frac{D_{ds}}{D_d D_s} \frac{2}{c^2} \int \Phi(\vec{r}) dz$$

- **Poisson equation:** $\nabla^2\Phi = 4\pi G\rho \Rightarrow \nabla^2\psi = 2\kappa$

- different surface mass distributions $\Rightarrow$ different deflection potentials $\Rightarrow$ different extended lens models

Circularly symmetric lens

- for a general mass distribution, the deflection angle is calculated using numerical integration
- analytical expressions could be obtained only for some simple mass distributions, such as spherically symmetric mass distribution (i.e. **axially symmetric** distribution of the projected mass)
- the deflection angle for axially symmetric lenses: $\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4GM(|\vec{\xi}|)}{c^2|\vec{\xi}|^2}\vec{\xi},$

  where $M(\xi)$ is the projected mass enclosed by the circle of radius: $\xi = |\vec{\xi}|$
- the same expression for deflection angle as in the case of point-mass lens due to **Birkhoff theorem**: gravitational force caused by a spherically-symmetric mass shell vanishes inside of it $\Rightarrow$ rings of projected axisymmetric mass cause no deflection inside of them
- like in the point-mass case, the lens equation is one-dimensional
- circularly symmetric lenses typically produce either 1 or 3 images, depending on the position of the source

Singular Isothermal Sphere (SIS)

- galaxies are often modeled by SIS since this is the simplest model that best reproduces their observed flat rotation curves
- analogy between stars and particles of an ideal gas in thermal equilibrium where the temperature is related to the one-dimensional **velocity dispersion** σ_v
- stellar gas is assumed to be **isothermal** across the galaxy: $\sigma_v = const$
- 3D density distribution: $\rho(r) = \frac{\sigma_\nu^2}{2\pi G}\frac{1}{r^2}$
- rotational velocity of stars in circular orbits: $v_{\rm rot}^2(r) = \frac{GM(r)}{r} = 2\sigma_v^2 = const$
- surface mass density: $\Sigma(\xi) = \int \rho(r)dz = \frac{\sigma_v^2}{2G}\frac{1}{\xi}$
- deflection angle: $M(\xi) = \int\limits_0^\xi \Sigma(\xi')2\pi\xi' d\xi' \Rightarrow \hat{\alpha}(\xi) = 4\pi\frac{\sigma_v^2}{c^2} = 1''.4\left(\frac{\sigma_v}{220\,{\rm km\,s^{-1}}}\right)^2$
- Einstein (critical) radius: $\theta_E = 4\pi\frac{\sigma_v^2}{c^2}\frac{D_{ds}}{D_s} = \hat{\alpha}\frac{D_{ds}}{D_s} = \alpha$

SIS lens model

$\alpha = \theta_E \Rightarrow \vec{\alpha} = \theta_E \frac{\vec{\theta}}{\theta}$ $\Rightarrow$ the **lens equation in the case of SIS lens**:

$$\vec{\beta} = \vec{\theta} - \theta_E \frac{\vec{\theta}}{\theta} \Bigg/ \cdot \vec{\theta} \quad \Leftrightarrow \quad \vec{y} = \vec{x} - \frac{\vec{x}}{x} \Bigg/ \cdot \vec{x}$$

(where the second form employs the position angles of the lens and source normalized to θ_E) $\Rightarrow$ the lens equation has two solutions:

$\theta_{\pm} = \beta \pm \theta_E \Leftrightarrow x = y \pm 1$

- for $\beta < \theta_E \Leftrightarrow y < 1$ there are two images on the opposite sides of the lens center, separated by $\Delta\theta = 2\theta_E$:

  1. $\theta_+ = \beta + \theta_E \Leftrightarrow x_+ = y + 1$
  2. $\theta_- = \beta - \theta_E \Leftrightarrow x_- = y - 1$

- for $\beta > \theta_E \Leftrightarrow y > 1$ only one image occurs at: $x_+ = y + 1$
- multiple images only if the source lies inside the Einstein ring: $\beta < \theta_E$

Deflection potentials of simple lens models

- **Point-like:** $\Sigma(\vec{\xi}) = M\delta_D(\vec{\xi}) \Rightarrow \psi(\vec{\theta}) = \theta_E^2 \ln|\vec{\theta}|$
- **Singular Isothermal Sphere (SIS):** $\psi(\vec{\theta}) = b \cdot |\vec{\theta}|, \quad b = 4\pi \left(\frac{\sigma}{c}\right)^2 \frac{D_{ds}}{D_s}$
- b - critical radius ($b = \theta_E$)
- **Softened Isothermal Sphere:**
- generalization of SIS model which is more realistic for (spiral) galaxies
- isothermal sphere with **finite core** of radius ξ_c
- deflection angle is modified to: $\hat{\alpha}(\xi) = 4\pi \frac{\sigma_v^2}{c^2} \frac{\xi}{\sqrt{\xi^2 + \xi_c^2}} \Rightarrow \psi(\theta) = b \cdot \sqrt{\theta^2 + \theta_c^2}$

Lens Model	$\psi(\theta)$	$\alpha(\theta)$
Point mass	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4GM}{D_{\rm d} c^2} \ln|\theta|$	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4GM}{c^2 D_{\rm d} |\theta|}$
Singular isothermal sphere	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} |\theta|$	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2}$
Softened isothermal sphere	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} \left(\theta_{\rm c}^2 + \theta^2\right)^{1/2}$	$\frac{D_{\rm ds}}{D_{\rm s}} \frac{4\pi\sigma^2}{c^2} \frac{\theta}{\left(\theta_{\rm c}^2 + \theta^2\right)^{1/2}}$

Isothermal Ellipsoid

- describes an elliptical galaxy lens
- a straightforward generalization of the isothermal sphere with finite core:

$$\Sigma(\theta_1, \theta_2) = \frac{\Sigma_0}{\left[\theta_c^2 + (1-\epsilon)\theta_1^2 + (1+\epsilon)\theta_2^2\right]^{1/2}} \ ,$$

  where θ_1, θ_2 are orthogonal coordinates along the major and minor axes of the lens measured from the center
- the potential $\psi(\theta_1, \theta_2)$ corresponding to above density distribution is somewhat complicated, and instead of the elliptical density model, it is simpler and often sufficient to model a galaxy by means of an elliptical effective lensing potential:

$$\psi(\theta_1, \theta_2) = \frac{D_{\rm ds}}{D_{\rm s}} 4\pi \frac{\sigma_v^2}{c^2} \left[\theta_c^2 + (1-\epsilon)\theta_1^2 + (1+\epsilon)\theta_2^2\right]^{1/2} \ ,$$

  where ε measures the ellipticity

Mass Models for Lensing

Model	N_r	Density $\rho(r)$	Surface Density $\kappa(r)$
Point mass	0	$\delta(\mathbf{x})$	$\delta(\mathbf{x})$
Power law or α-models	2	$\left(s^2+r^2\right)^{(\alpha-3)/2}$	$\left(s^2+r^2\right)^{(\alpha-2)/2}$
Isothermal ($\alpha=1$)	1	$\left(s^2+r^2\right)^{-1}$	$\left(s^2+r^2\right)^{-1/2}$
$\alpha=-1$	1	$\left(s^2+r^2\right)^{-2}$	$\left(s^2+r^2\right)^{-3/2}$
Pseudo-Jaffe	2	$\left(s^2+r^2\right)^{-1}\left(a^2+r^2\right)^{-1}$	$\left(s^2+r^2\right)^{-1/2}-\left(a^2+r^2\right)^{-1/2}$
King (approximate)	1	$\cdots$	$2.12\left(0.75r_s^2+r^2\right)^{-1/2}$ $-1.75\left(2.99r_s^2+r^2\right)^{-1/2}$
de Vaucouleurs	1	$\cdots$	$\exp\left[-7.67(r/R_e)^{1/4}\right]$
Hernquist	1	$r^{-1}\left(r_s+r\right)^{-3}$	see eq. (47)
NFW	1	$r^{-1}\left(r_s+r\right)^{-2}$	see eq. (53)
Cuspy NFW	2	$r^{-\gamma}\left(r_s+r\right)^{\gamma-3}$	see eq. (57)
Cusp	3	$r^{-\gamma}\left(r_s^2+r^2\right)^{(\gamma-n)/2}$	see eq. (64)
Nuker	4	$\cdots$	see eq. (71)
Exponential disk	1	$\cdots$	$\exp[-r/R_d]$
Kuzmin disk	1	$\cdots$	$\left(r_s^2+r^2\right)^{-3/2}$

Exam questions

1. Extended lenses: surface mass density and convergence
2. Deflection potential and isothermal lens models

Literature

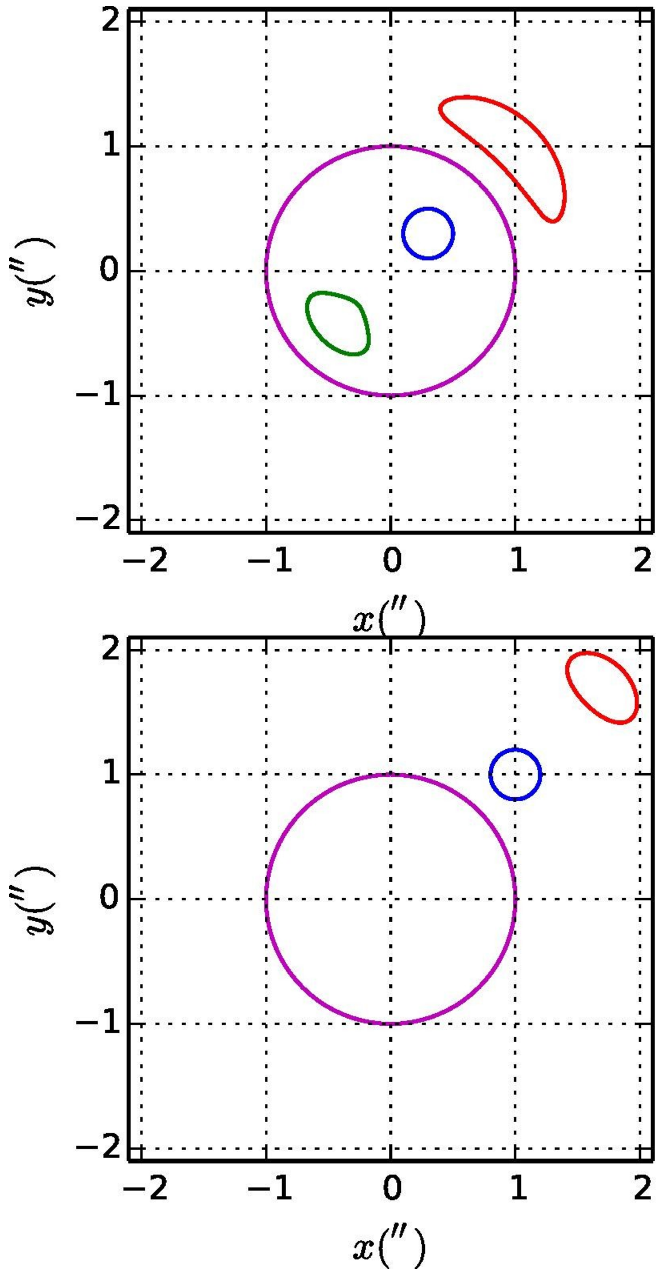
Textbook:

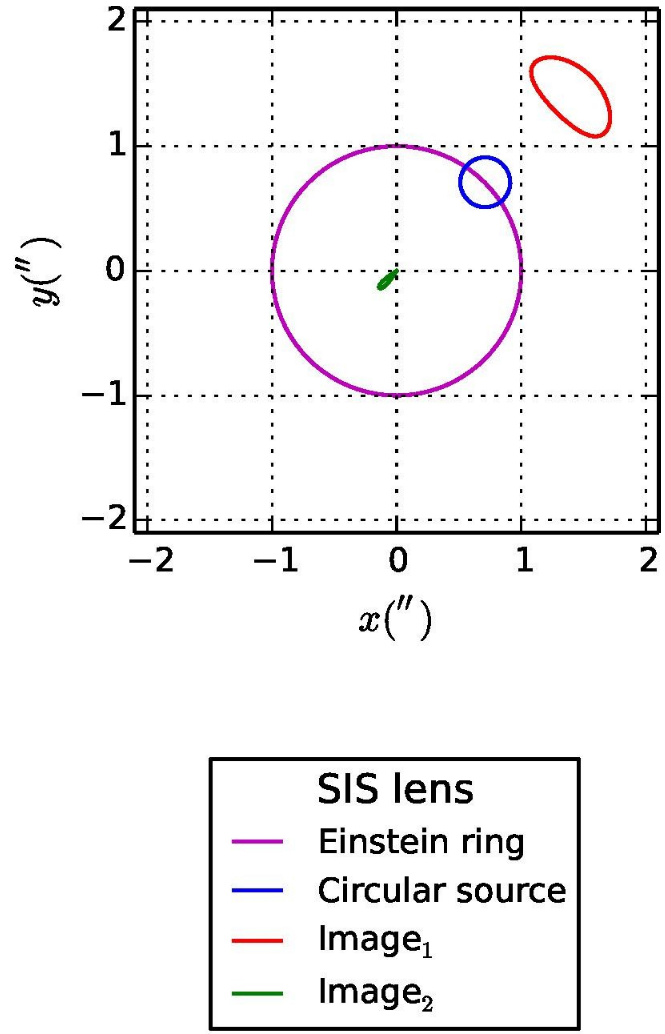
- *Gravitational Lensing*: Strong, Weak and Micro, Book Series: Saas-Fee Advanced Courses
  1. P. Schneider - *Introduction to Gravitational Lensing and Cosmology*
  2. C. S. Kochanek - *Strong Gravitational Lensing*
  3. P. Schneider - *Weak Gravitational Lensing*
  4. J. Wambsganss - *Gravitational Microlensing*

Exercise 1

Consider a SIS gravitational lens with Einstein radius of 1" located at coordinate origin, and calculate the images of a circular background source with radius of 0".2, assuming that it is located at the following three positions: inside the Einstein ring at (0".3, 0".3),
exactly at Einstein ring at (0".71, 0".71) and outside the Einstein ring at (1", 1"). Discuss the obtained results.

Solution 1





Solution is obtained by Python script "sisl_circs.py"