MASS 2023 Course:

Gravitational Lenses

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Lecture 09

1. Shear
   - Shear approximation of lensing potential
   - Simple SIS models with shear
   - Lens mapping
   - Distortion matrix
   - Magnification of images
   - **Critical curves and caustics**
2. Exercise

Shear approximation of lensing potential I

- Deflection (lensing) potential:

$$\psi(\vec{\theta}) = \frac{1}{\pi}\int_{\mathbb{R}^2} \kappa(\vec{\theta}')\ln|\vec{\theta}-\vec{\theta}'|d^2\theta' \quad \Rightarrow \quad \vec{\alpha} = \vec{\nabla}\psi(\vec{\theta}) \quad \Rightarrow \quad \vec{\beta} = \vec{\theta} - \vec{\nabla}\psi(\vec{\theta})$$

- Expansion of the deflection potential in a Taylor series:

$$\psi(x,y) = \psi_0 + \psi_x x + \psi_y y + \frac{1}{2}\left(\psi_{xx}x^2 + \psi_{yy}y^2\right) + \psi_{xy}xy + \ldots$$

- $\psi_0 = 0$ since constant terms in the potential have no effect on lensing observables
- The linear terms in the expansion correspond to constant terms in the reduced deflection angle
- **Definitions:**

$$\kappa \equiv \frac{1}{2}(\psi_{xx}+\psi_{yy}), \quad \gamma_1 \equiv \frac{1}{2}(\psi_{xx}-\psi_{yy}) \quad \text{and} \quad \gamma_2 \equiv \psi_{xy}$$

- ***κ* - convergence:** gives rise to isotropic magnification
- ***γ* - shear:** describes the quadrupole term of the perturbing potential

Shear approximation of lensing potential II

- The potential is then:

$$\psi(x,y) = \frac{1}{2}\left[(\kappa+\gamma_1)x^2 + (\kappa-\gamma_1)y^2\right] + \gamma_2 xy$$
$$= \frac{r^2}{2}\left[\kappa + \gamma\cos 2(\theta-\theta_\gamma)\right],$$

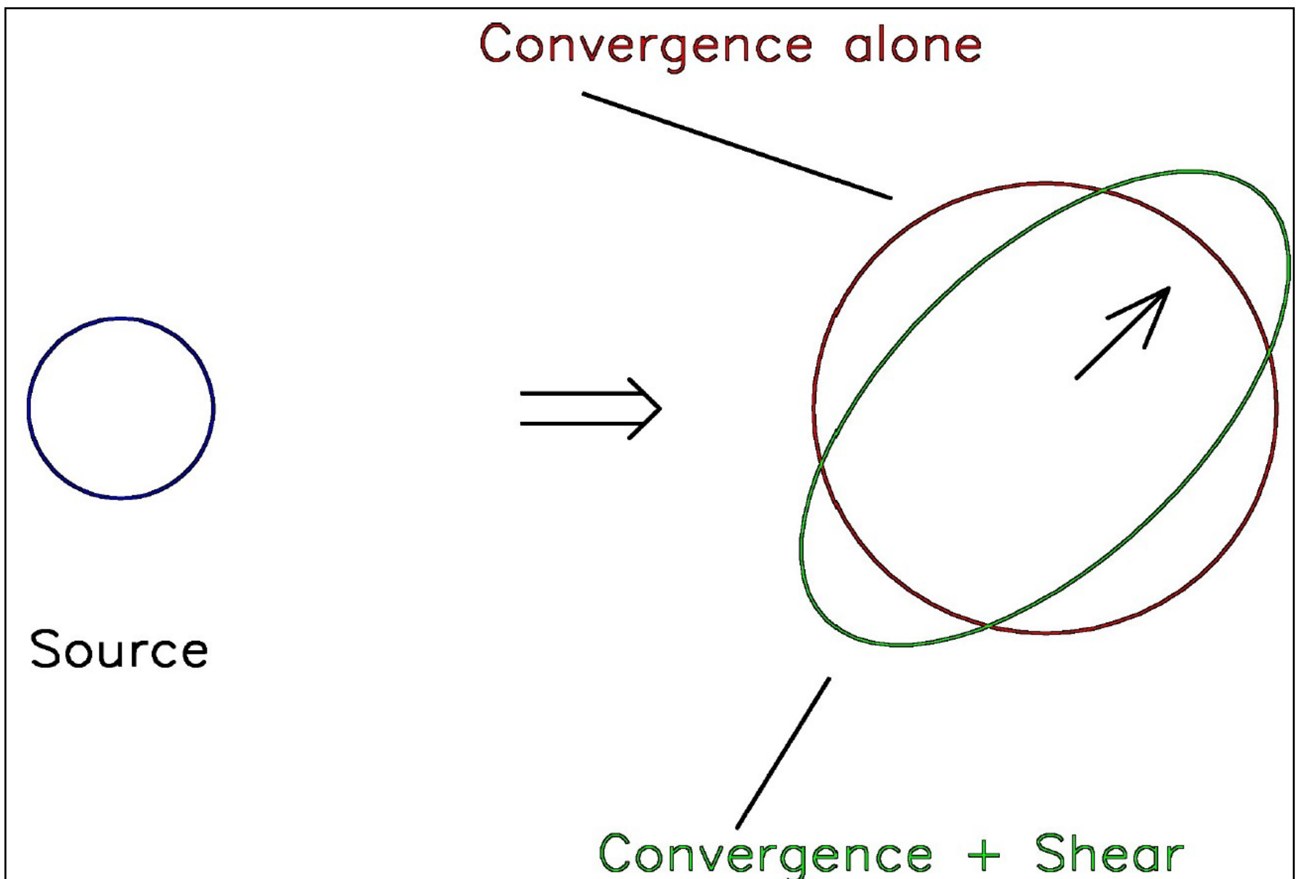
where $\quad (x,y) \equiv (r\cos\theta, r\sin\theta)$

and $\quad (\gamma\cos 2\theta_\gamma, \gamma\sin 2\theta_\gamma) \equiv (\gamma_1, \gamma_2)$

- γ and θ_γ are the amplitude and direction of the shear
- γ_1 and γ_2 are the components of the shear
- angle θ_γ reflects the direction of the shear-inducing tidal force relative to the coordinate system

Image distortions due to lens mapping

- The lensed images are distorted in shape and size
- Size distortion is due to the convergence κ: isotropic magnification
- Shape distortion is due to the tidal gravitational field described by the shear γ



Simple SIS models with shear

Shear types:

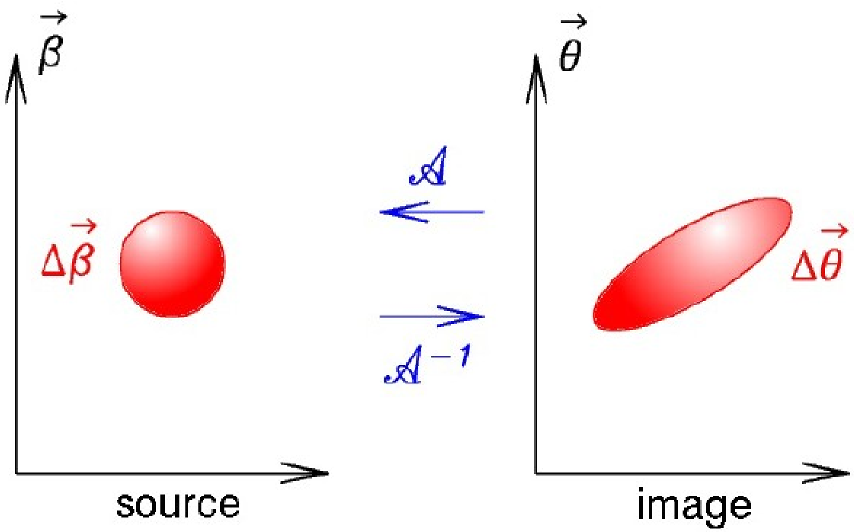
- **internal:** generated by density distribution fully inside the ring of images
- **external:** generated by density distribution fully outside the ring of images
- **mixed:** contributions from both inside and outside the ring of images
- **SIS:** $\psi(r, \theta) = b\, r$, where b is critical radius
- SIS models with different distribution of mass in the lens, which can produce four images:

$$\text{Model 1}: \quad br - \frac{\gamma}{2}\frac{b^4}{r^2}\cos 2(\theta - \theta_\gamma) \quad \text{SIS + Internal}$$

$$\text{Model 2}: \quad br + \gamma br \cos 2(\theta - \theta_\gamma) \quad \text{SIS + Mixed}$$

$$\text{Model 3}: \quad br + \frac{\gamma}{2} r^2 \cos 2(\theta - \theta_\gamma) \quad \text{SIS + External}$$

Lens mapping



- *local* imaging properties are described by Jacobian:

$$\mathcal{A} \equiv \frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \mathcal{I} - \frac{\partial\vec{\alpha}}{\partial\vec{\theta}}$$

- small sources correspond to images according to

$$\delta\vec{\theta} = \mathcal{A}^{-1}\delta\vec{\beta}$$

- circular source with radius r becomes ellipse with semi-major and semi-minor axes

$$a = \frac{r}{1-\kappa-\gamma}, \quad b = \frac{r}{1-\kappa+\gamma}$$

- *local* magnification

$$\mu \equiv (\det\mathcal{A})^{-1}$$

Distortion matrix

- In the vicinity of an arbitrary point, the lens equation represents mapping which can be described by its Jacobian matrix:

$$\mathcal{A} = \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$$

- The Jacobian matrix can be then written as:

$$\mathcal{A} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

- Necessary and sufficient conditions for multiple imaging:

  1. An isolated lens can produce multiple images if, and only if, there is a point with $det\ A < 0$
  2. A sufficient (but not necessary) condition for possible multiple images is that there exists a point such that $\kappa > 1$

- Local magnification: $\mu = (\det \mathcal{A})^{-1} = \dfrac{1}{(1-\kappa)^2 - \gamma^2}$

Magnification of images

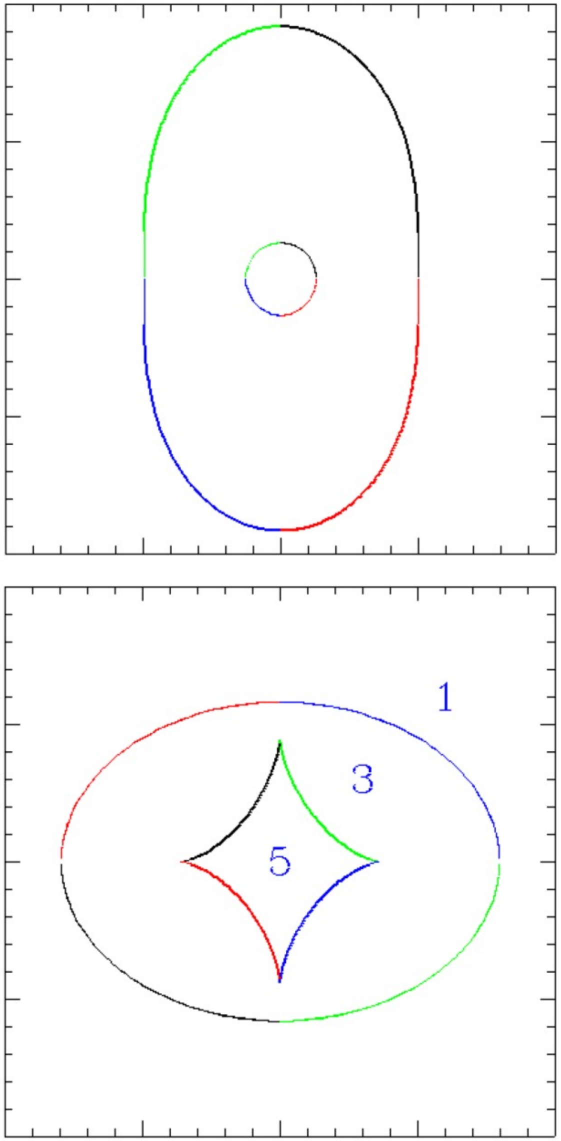
- The inverse magnification of a particular image i:

$$\mu_i^{-1} = \left(1 - \frac{\partial^2 \psi}{\partial x^2}\right)\left(1 - \frac{\partial^2 \psi}{\partial y^2}\right) - \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2$$

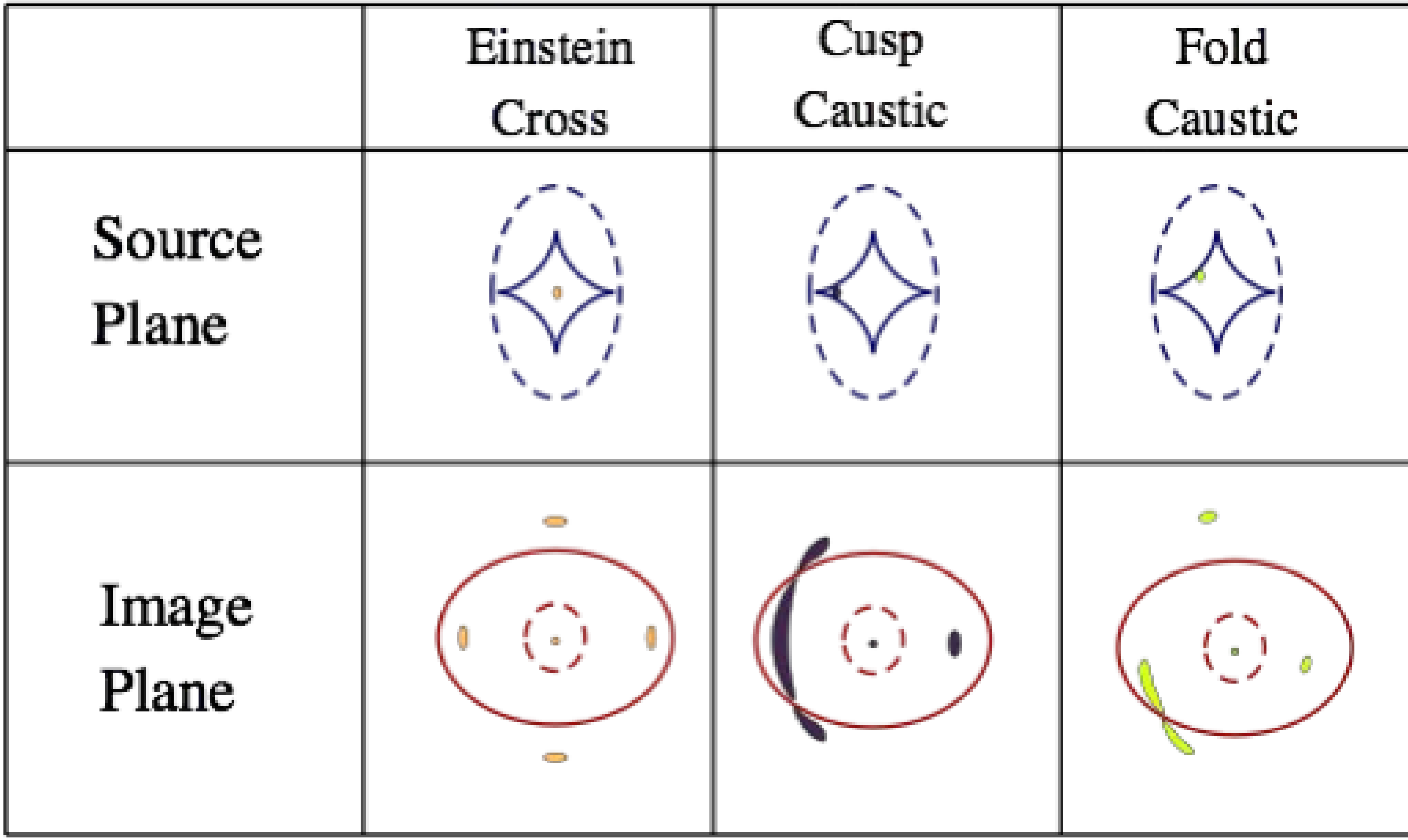
- Magnification is caused by both isotropic focusing due to the local matter density κ and anisotropic focusing due to shear γ
- The sign of μ is called the **parity** of an image
- Negative-parity images are mirror-symmetric images of the source
- Observed fluxes of images are determined by the absolute value of μ
- Since the intrinsic luminosity of sources is unknown, the magnification in a lens system is not an observable
- However, the flux ratio of different images provides a direct measurement of the absolute value of the corresponding magnification ratio

Critical curves and caustics

- locations in the lens plane at which *det A* = 0 have formally infinite magnification and are called **critical curves**
- the corresponding locations in the source plane are the **caustics**
- point-like lens: critical curve is its Einstein ring and caustic is the point: $\vec{\beta} = 0$
- spherically symmetric lenses: critical curves are circles
- elliptical or spherically symmetric lenses with external shear: caustics can consist of **cusps** and **folds**
- importance for number and parity of the images



The critical curves (upper panel) and caustics (lower panel) for an elliptical lens. The numbers identify regions in the source plane that correspond to 1, 3 or 5 images, respectively. The smooth lines are called **fold caustics**, and the tips at which two fold caustics connect are called **cusp caustics**



The images due to lensing from a non-singular isothermal ellipsoid

Tangential and radial critical curves and caustics

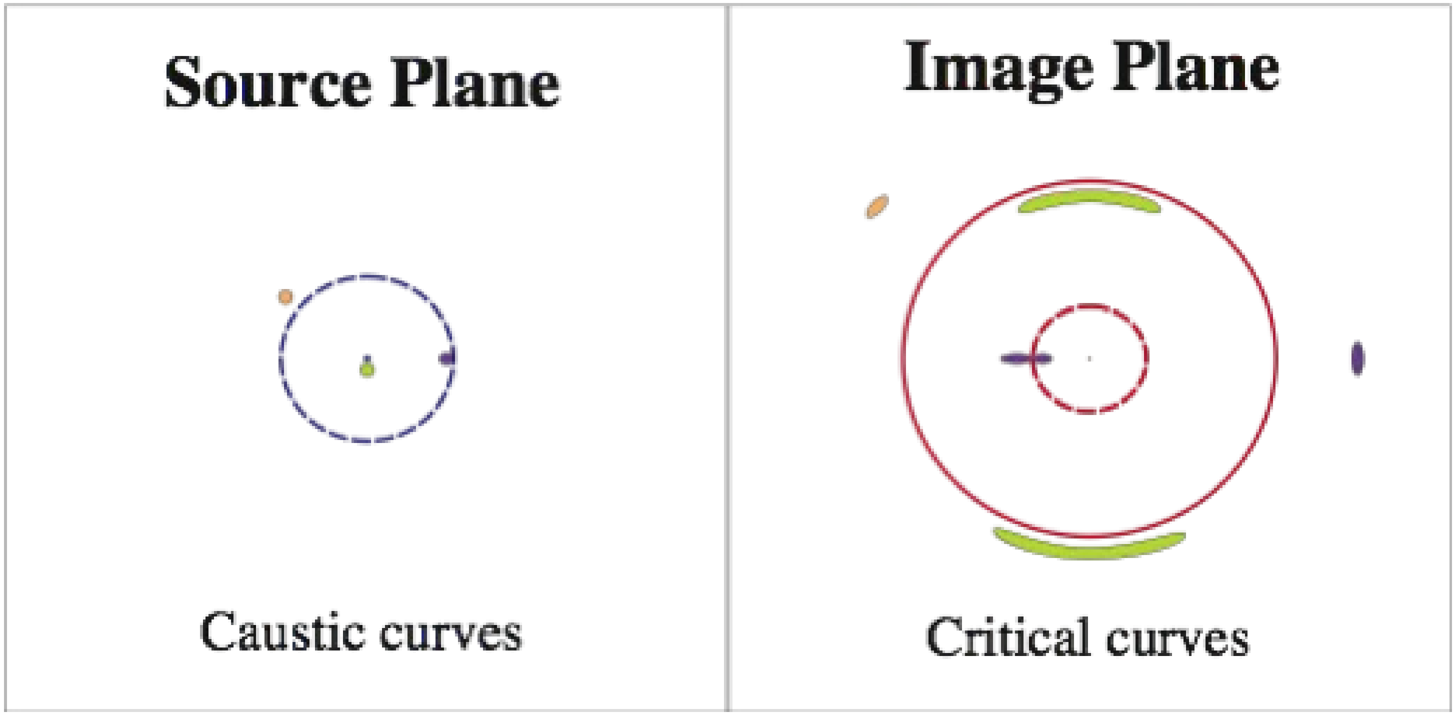
$$\mu \equiv \det M = \frac{1}{\det A} = \frac{1}{(1-\kappa)^2 - \gamma^2}$$

The eigenvalues of the magnification tensor (or the inverse eigenvalues of the Jacobian matrix) measure the amplification in the tangential and in the radial direction and are given by

$$\mu_t = \frac{1}{\lambda_t} = \frac{1}{1-\kappa-\gamma}$$

$$\mu_r = \frac{1}{\lambda_r} = \frac{1}{1-\kappa+\gamma}.$$

The magnification is ideally infinite where $\lambda_t = 0$ and where $\lambda_r = 0$. These two conditions define two curves in the lens plane, called the *tangential* and the *radial critical line*, respectively. An image forming along the tangential critical line is strongly distorted tangentially to this line. On the other hand, an image forming close to the radial critical line is stretched in the direction perpendicular to the line itself.



The images formed by three sources on the source plane due to gravitational lensing by a non-singular Isothermal sphere. The number of images formed by a source depends on its position relative to the caustic curves

- **<u>In the case of SIS lenses:</u>**

1. **radial (pseudo)-caustic:** curve in the source plane associated with a **radial critical curve** at the origin in the lens plane ($x = 0, y = 0$)
   - pseudo-caustic because there are neither images nor a divergent magnification associated with it
2. **tangential caustic**: the image of the tangential critical curve on the source plane (astroid)

Exam questions

1. Shear (definition and types) and simple SIS lens models with shear
2. Distortion matrix, local magnification, critical curves and caustics

Literature

Textbook:

- *Gravitational Lensing*: Strong, Weak and Micro, Book Series: Saas-Fee Advanced Courses
  1. P. Schneider - *Introduction to Gravitational Lensing and Cosmology*
  2. C. S. Kochanek - *Strong Gravitational Lensing*
  3. P. Schneider - *Weak Gravitational Lensing*
  4. J. Wambsganss - *Gravitational Microlensing*

Exercise 1

Assume a point-like source located at (0".07, 0".0), which is lensed by a SIS lens with critical radius b = 0".7 and external shear with amplitude $\gamma = 0.1$ and direction $\theta_\gamma = 0$, and calculate:

a) the image positions and magnifications
b) the critical curves (plot them together with the image positions and discuss the situation in the lens plane)
c) the caustics (plot them together with the source position and discuss the situation in the source plane)

Solution 1

SIS + XS ($\theta_\gamma = 0$): $\psi(r,\theta) = b\,r + \frac{\gamma}{2} r^2 \cos 2\theta = b\,r + \frac{\gamma}{2} r^2 \left(\cos^2\theta - \sin^2\theta\right)$

$x = r\cos\theta,\ \ y = r\sin\theta \ \Rightarrow\ \psi(x,y) = b\sqrt{x^2+y^2} + \frac{\gamma}{2}\left(x^2 - y^2\right)$

$$\frac{\partial\psi}{\partial x} = \frac{b\,x}{\sqrt{x^2+y^2}} + \gamma x, \quad \frac{\partial\psi}{\partial y} = \frac{b\,y}{\sqrt{x^2+y^2}} - \gamma y$$

lens equation: $\vec{u} = \vec{x} - \vec{\nabla}\,\psi(\vec{x}) \ \underset{\vec{x}=(x,y)}{\overset{\vec{u}=u(u,v)}{\Rightarrow}} \ \begin{cases} u = (1-\gamma)\,x - \dfrac{b\,x}{\sqrt{x^2+y^2}} \\ v = (1+\gamma)\,y - \dfrac{b\,y}{\sqrt{x^2+y^2}} \end{cases}$

$$v = 0 \ \Rightarrow\ (1+\gamma)\,y = \frac{b\,y}{\sqrt{x^2+y^2}} \ \Rightarrow\ \sqrt{x^2+y^2} = \frac{b}{1+\gamma} \ \vee\ y = 0$$

Image positions

$$u=(1-\gamma)x-\frac{b\,x}{\sqrt{x^2+y^2}} \overset{\sqrt{x^2+y^2}=\frac{b}{1+\gamma}}{\Rightarrow} x=-\frac{u}{2\,\gamma}$$

$$\sqrt{x^2+y^2}=\frac{b}{1+\gamma} \Rightarrow y=\pm\sqrt{\frac{b^2}{(1+\gamma)^2}-\frac{u^2}{4\,\gamma^2}}$$

- Image 1: (-0".35, 0".53)
- Image 2: (-0".35 , -0".53)

$$u=(1-\gamma)x-\frac{b\,x}{\sqrt{x^2+y^2}} \overset{y=0}{\Rightarrow} u=(1-\gamma)x-\frac{bx}{|x|} \Rightarrow x=\frac{u\pm b}{1-\gamma}$$

- Image 3: (0".86, 0".0)
- Image 4: (-0".70 , 0".0)

Image magnification

$$\frac{\partial^2\psi}{\partial x^2} = \frac{by^2}{\sqrt{\left(x^2+y^2\right)^3}} + \gamma, \quad \frac{\partial^2\psi}{\partial y^2} = \frac{bx^2}{\sqrt{\left(x^2+y^2\right)^3}} - \gamma, \quad \frac{\partial^2\psi}{\partial x \partial y} = -\frac{bxy}{\sqrt{\left(x^2+y^2\right)^3}}$$

$$\mu_i^{-1} = \left(1 - \frac{\partial^2\psi}{\partial x^2}\right)\left(1 - \frac{\partial^2\psi}{\partial y^2}\right) - \left(\frac{\partial^2\psi}{\partial x \partial y}\right)^2 = 1 - \gamma^2 - \frac{b}{\sqrt{x^2+y^2}}\left(1 - \gamma\frac{x^2-y^2}{x^2+y^2}\right) \Rightarrow$$

Image 1: μ_1 = -6.44,

Image 2: μ_2 = -6.44,

Image 3: μ_3 = 3.88,

Image 4: μ_4 = 11.11

Critical curves and caustics

- **radial critical curve:** $x = 0, y = 0$
- **radial (pseudo)-caustic:**

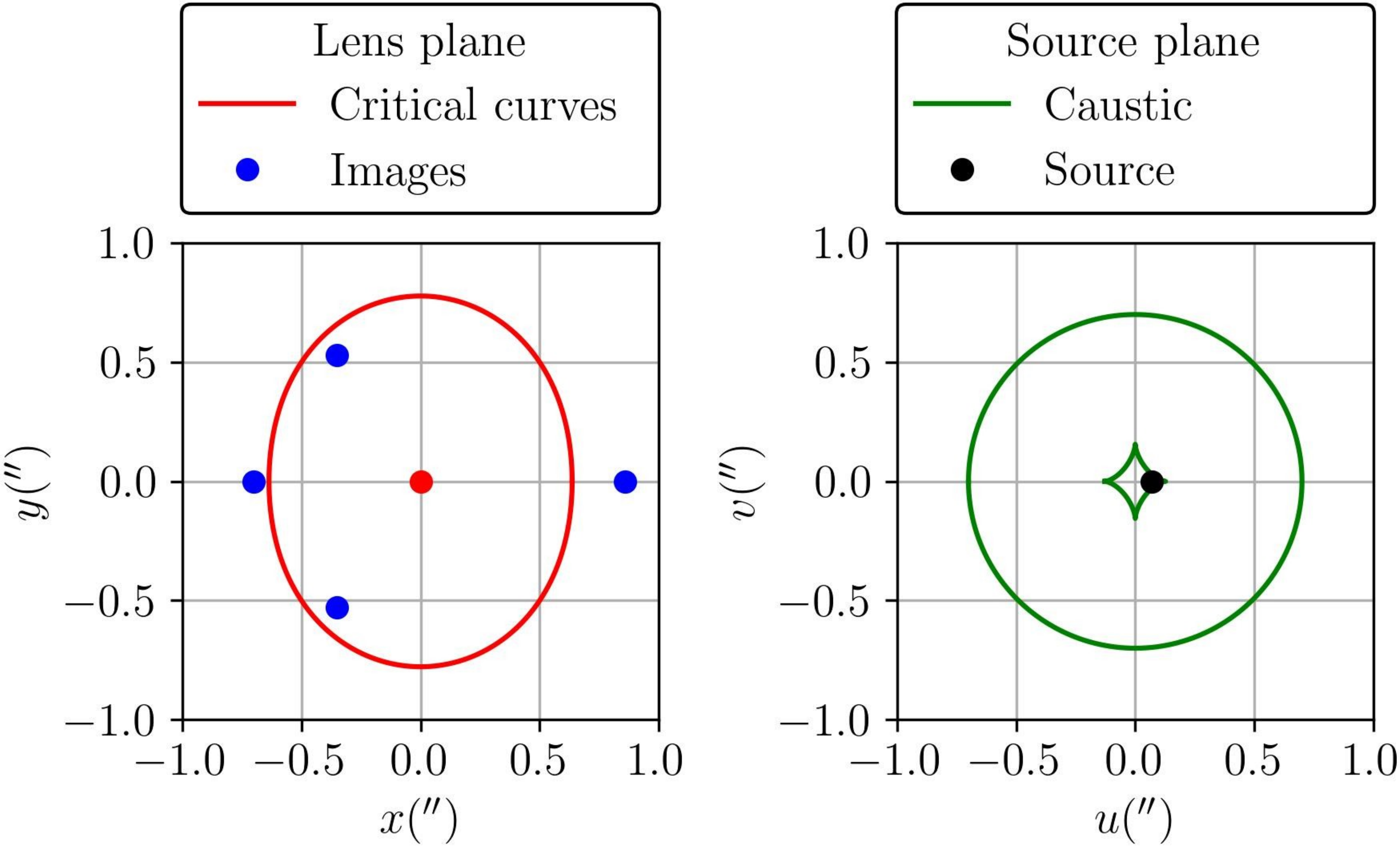
$$\left.\begin{aligned} u &= (1-\gamma)x - \frac{bx}{\sqrt{x^2+y^2}} = (1-\gamma)r\cos\theta - b\cos\theta \\ v &= (1+\gamma)y - \frac{by}{\sqrt{x^2+y^2}} = (1+\gamma)r\sin\theta - b\sin\theta \end{aligned}\right\} \overset{r=0}{\Rightarrow} \begin{aligned} u &= -b\cos\theta \\ v &= -b\sin\theta \end{aligned}$$

- μ^{-1} in polar coordinates:

$$\mu^{-1} = 1-\gamma^2 - \frac{b}{r}(1-\gamma\cos 2\theta) = 0 \Rightarrow r = b\frac{1-\gamma\cos 2\theta}{1-\gamma^2}$$

- **tangential critical curve:**
- **tangential caustic**:

$$\left.\begin{aligned} x &= r\cos\theta = b\cos\theta\frac{1-\gamma\cos 2\theta}{1-\gamma^2} \\ y &= r\sin\theta = b\sin\theta\frac{1-\gamma\cos 2\theta}{1-\gamma^2} \end{aligned}\right\} \overset{\text{lens eq.}}{\Rightarrow} \left\{\begin{aligned} u &= -\frac{2b\gamma}{1+\gamma}\cos^3\theta \\ v &= \frac{2b\gamma}{1-\gamma}\sin^3\theta \end{aligned}\right.$$



Solution is obtained by Python script "SIS_XS.py"